WEAK FACTORIZATIONS, FRACTIONS AND HOMOTOPIES

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ABSTRACT. We show that the homotopy category can be assigned to any category equipped with a weak factorization system. A classical example of this construction is the stable category of modules. We discuss a connection with the open map approach to bisimulations proposed by Joyal, Nielsen and Winskel.

1. Introduction

Weak factorization systems originated in homotopy theory (see [Q], [Bo], [Be] and [AHRT]). Having a weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$, we can formally invert the morphisms from $\mathcal{R}$ and form the category of fractions $\mathcal{K}[\mathcal{R}^{-1}]$. From the point of view of homotopy theory, we invert too few morphisms: only trivial fibrations and not all weak equivalences. Our aim is to show that this procedure is important in many situations.

For instance, the class *Mono* of all monomorphisms form a left part of the weak factorization system $(Mono, \mathcal{R})$ in a category R-**Mod** of (left) modules over a ring R. Then R-**Mod** $[\mathcal{R}^{-1}]$ is the usual stable category of modules. Or, in the open map approach to bisimulations suggested in [JNW], one considers a weak factorization system $(\mathcal{L}, \mathcal{O}_\mathcal{P})$, where $\mathcal{O}_\mathcal{P}$ is the class of $\mathcal{P}$-open morphisms w.r.t. a given full subcategory $\mathcal{P}$ of path objects. Then two objects K and L are $\mathcal{P}$-bisimilar iff there is a span

$$K \xleftarrow{f} M \xrightarrow{g} L$$

of $\mathcal{P}$-open morphisms. Any two $\mathcal{P}$-bisimilar objects are isomorphic in the fraction category $\mathcal{K}[\mathcal{O}_\mathcal{P}^{-1}]$ but, in general, the fraction category makes more objects isomorphic than just $\mathcal{P}$-bisimilar ones.

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Any weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ with finite coproducts yields a cylinder object in $\mathcal{K}$ and thus a relation $\sim$ of homotopy between morphisms of $\mathcal{K}$. We will show that any two homotopic morphisms have the same image in the fraction category. Moreover, if $\mathcal{K}$ has finite coproducts and any morphism in $\mathcal{R}$ is a split epimorphism then the categories $\mathcal{K}[\mathcal{R}^{-1}]$ and $\mathcal{K}/\sim$ are equivalent.

2. Weak factorization systems

Definition 2.1. Let $\mathcal{K}$ be a category and $f : A \to B$, $g : C \to D$ morphisms such that in each commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

there is a diagonal $d : B \to C$ with $d \cdot f = u$ and $g \cdot d = v$. Then we say that g has the *right lifting property* w.r.t. f and f has the *left lifting property* w.r.t. g.

For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$\mathcal{H}^\square = \{g|g$ has the right lifting property w.r.t. each $f \in \mathcal{H}\}$ and

${}^\square\mathcal{H} = \{f|f$ has the left lifting property w.r.t. each $g \in \mathcal{H}\}$.

Definition 2.2. ([Be]) A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$ such that
(1) $\mathcal{R} = \mathcal{L}^\square$, $\mathcal{L} = {}^\square\mathcal{R}$ and
(2) any morphism h of $\mathcal{K}$ has a factorization $h = g \cdot f$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

The category of fractions $\mathcal{K}[\mathcal{S}^{-1}]$, where $\mathcal{K}$ is a category and $\mathcal{S}$ a class of morphisms in $\mathcal{K}$, was introduced in [GZ] (see [Bor], [KP]). This category has the same objects as $\mathcal{K}$ and is equipped with a functor $P : \mathcal{K} \to \mathcal{K}[\mathcal{S}^{-1}]$ sending any morphism from $\mathcal{S}$ to an isomorphism. Moreover, for every functor $F : \mathcal{K} \to \mathcal{X}$ sending any morphism from $\mathcal{S}$ to an isomorphism, there is a unique functor $\overline{F} : \mathcal{K}[\mathcal{S}^{-1}] \to \mathcal{X}$ such that $F = \overline{F} \cdot P$. It may happen that the category $\mathcal{K}[\mathcal{S}^{-1}]$ is not locally small, i.e., that one can have a proper class of morphisms between two given objects in $\mathcal{K}[\mathcal{S}^{-1}]$.

Observation 2.3. $\mathcal{K}[\mathcal{S}^{-1}]$ is a quotient of the category of zig-zags

$$\begin{tikzcd} K \arrow[r, "f_1"] & X_1 & X_2 \arrow[l, "f_2"'] \arrow[r] & \dots & L \end{tikzcd}$$

where all morphisms going backwards are in $\mathcal{S}$ (see [KP], II.2). If $\mathcal{K}$ has finite limits and $\mathcal{S}$ contains all isomorphisms, is closed under compositions and stable under pullbacks then these zig-zags can be reduced to spans

$$\begin{tikzcd} K & X \arrow[l, "s"'] \arrow[r, "f"] & L \end{tikzcd}$$

with $s \in \mathcal{S}$. In fact, a zig-zag is reduced to a span by means of pullbacks as follows:

$$\begin{tikzcd} K \arrow[r, "f_1"] & X_1 & X_2 \arrow[l, "f_2"'] \arrow[r] & \cdots & & L \\ & \overline{X_1} \arrow[ul, "\overline{f}_2"] \arrow[ur, "\overline{f}_1"'] & & & & \end{tikzcd}$$

The equivalence relation on spans giving the category of fractions is easy to describe if $\mathcal{S}$ also has the property: if $t \cdot f = t \cdot g$ with $t \in \mathcal{S}$ then $f \cdot s = g \cdot s$ for some $s \in \mathcal{S}$. One then says that $\mathcal{S}$ admits a right calculus of fractions (see [Bor]).

Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$ having finite limits. Then $\mathcal{R}$ contains all isomorphisms, is closed under compositions and stable under pullbacks (see [AHRT]). Hence the fraction category $\mathcal{K}[\mathcal{R}^{-1}]$ is a quotient category of the category of spans. But $\mathcal{R}$ rarely admits a right calculus of fractions.

Example 2.4. Let $(\mathcal{L}, \mathcal{R})$ be a factorization system in a category $\mathcal{K}$ having finite limits. This means that $(\mathcal{L}, \mathcal{R})$ is defined by means of a unique diagonalization property, i.e., that d in 2.1 is unique. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system (see 14.6 (3) in [AHS]). We show that $\mathcal{R}$ admits a right calculus of fractions. Consider

$$\begin{tikzcd} K \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & L \arrow[r, "t"] & M \end{tikzcd}$$

such that $t \cdot h_1 = t \cdot h_2$ and $t \in \mathcal{R}$. It suffices to show that the equalizer e of h_1 and h_2 belongs to $\mathcal{R}$. This means that we have to show that e has the unique diagonalization property w.r.t. each morphism $f \in \mathcal{L}$. Consider a commutative square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & N \arrow[d, "e"] \\ Y \arrow[r, "v"'] & K \end{tikzcd}$$

with $f \in \mathcal{S}$. Since

$$\begin{tikzcd}
X \arrow[r, "h_1 eu"] \arrow[d, "f"'] & L \arrow[d, "t"] \\
Y \arrow[ur, shift left, "h_1 v"] \arrow[ur, shift right, "h_2 v"'] \arrow[r, "th_1 v"'] & M
\end{tikzcd}$$

commutes, the unique diagonalization property yields that $h_1 \cdot v = h_2 \cdot v$. Thus $v = w \cdot e$ for some $w : Y \to N$. Since e is a monomorphism, w is the unique diagonal in the starting square.

In the case of the factorization system $(\mathrm{Iso}(\mathcal{K}), \mathrm{Mor}(\mathcal{K}))$, the fraction category $\mathcal{K}[\mathrm{Mor}(\mathcal{K})^{-1}]$ is the set of connected components of $\mathcal{K}$. Here, $\mathrm{Iso}(\mathcal{K})$ consists of isomorphisms of $\mathcal{K}$ and $\mathrm{Mor}(\mathcal{K})$ of all morphisms of $\mathcal{K}$.

Observation 2.5. The class

$$\overline{\mathcal{S}} = \{f \mid P(f) \text{ is an isomorphism}\}$$

is called the *saturation* of $\mathcal{S}$. It is easy to see that $\overline{\mathcal{S}}$ is closed under retracts in the arrow category $\mathcal{K}^{\to}$ and has the 2-out-of-3 property, i.e., with any two of f, g, $g \cdot f$ belonging to $\overline{\mathcal{S}}$ also the third morphism belongs to $\overline{\mathcal{S}}$.

The following definition is motivated by [JNW].

Definition 2.6. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$. Two objects K and L are called *bisimilar* if there is a span

$$K \xleftarrow{f} X \xrightarrow{g} L$$

with $f, g \in \mathcal{R}$.

Observation 2.7. Any two bisimilar objects are clearly isomorphic in the fraction category $\mathcal{K}[\mathcal{R}^{-1}]$. If $\mathcal{K}$ has finite limits then bisimilarity is an equivalence relation. We will see later that (see 3.6), even in this case, two objects K, L may be isomorphic in $\mathcal{K}[\mathcal{R}^{-1}]$ without being bisimilar.

Observation 2.8. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$ having an initial object 0. Then the following two conditions are equivalent:

(1) any morphism from $\mathcal{R}$ is a split epimorphism,

(2) any morphism $0 \to K$ belongs to $\mathcal{L}$.

Indeed, (1) ⇒ (2) because a diagonal in a square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & L \arrow[d, "f"] \\ K \arrow[r, "v"'] & M \end{tikzcd}$$

is $t \cdot v$ where t splits $f \in \mathcal{R}$. Conversely, (2) ⇒ (1) because a diagonal in a square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & L \arrow[d, "f"] \\ K \arrow[r, "\mathrm{id}_K"'] & K \end{tikzcd}$$

splits $f \in \mathcal{R}$.

Observation 2.9. Let $\mathcal{K}$ be a locally presentable category (cf. [AR]) and $\mathcal{C}$ a set of morphisms in $\mathcal{K}$. Then $\left({}^{\square}(\mathcal{C}^{\square}), \mathcal{C}^{\square}\right)$ is a weak factorization system and ${}^{\square}(\mathcal{C}^{\square})$ is the smallest class $\mathcal{L}$ containing $\mathcal{C}$ which is

(a) closed under retracts in $\mathcal{K}^{\to}$,
(b) closed under compositions and contains all isomorphisms,
(c) stable under pushouts,
(d) closed under transfinite compositions, i.e., given a smooth chain of morphisms $(f_{ij} : K_i \to K_j)_{i<j<\lambda}$ from $\mathcal{L}$ (i.e., λ is a limit ordinal, $f_{jk} \cdot f_{ij} = f_{ik}$ for $i < j < k$ and $f_{ij} : K_i \to K_j$ is a colimit cocone for any limit ordinal $j < \lambda$), then a colimit cocone $f_i : K_i \to K$ has $f_0 \in \mathcal{L}$.

(see [Be] or [AHRT]). Even, ${}^{\square}(\mathcal{C}^{\square})$ consists of retracts of transfinite compositions of pushouts of morphisms from $\mathcal{C}$.

3. Homotopies

Definition 3.1. Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. For an object K of $\mathcal{K}$, we get a *cylinder object* $\overline{K}$ by a factorization of the codiagonal

$$\nabla : K + K \xrightarrow{c_K} \overline{K} \xrightarrow{s_K} K$$

with $c_K \in \mathcal{L}$ and $s_K \in \mathcal{R}$. We denote by

$$c_K^1 = c_K \cdot i_1 \quad \text{and} \quad c_K^2 = c_K \cdot i_2$$

the compositions of c_K with the coproduct injections

$$K \xrightarrow{i_1} K + K \xleftarrow{i_2} K\,.$$

There is a well-known way of getting homotopy from cylinder objects (see [KP]). Having two morphisms $f, g : K \to L$, we say that f and g are homotopic and write $f \sim g$ if there is a morphism $h : \overline{K} \to L$ such that the following diagram commutes

$$\begin{tikzcd} K+K \arrow[rr, "(f{,}g)"] \arrow[dr, "c_K"'] & & L \\ & \overline{K} \arrow[ur, "h"'] & \end{tikzcd}$$

Here, $(f, g)\cdot i_1 = f$ and $(f, g)\cdot i_2 = g$. The homotopy relation $\sim$ is clearly reflexive and symmetric. But, in general, the homotopy relation is not transitive. On the other hand, $\sim$ is compatible with the composition.

Lemma 3.2. *Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. Let $f, g : K \to L$, $u : L \to M$ and $v : N \to K$ be in $\mathcal{K}$ and $f \sim g$. Then $u \cdot f \sim u \cdot g$ and $f \cdot v \sim g \cdot v$.*

Proof. Let $h : \overline{K} \to L$ make f and g homotopic. Then $u \cdot h$ makes $u \cdot f$ and $u \cdot g$ homotopic. Using a lifting property, there is a morphism t such that both squares in the following diagram are commutative

$$\begin{tikzcd} N+N \arrow[r, "v+v"] \arrow[d, "c_N"'] & K+K \arrow[d, "c_K"] \\ \overline{N} \arrow[r, dotted, "t"] \arrow[d, "s_N"'] & \overline{K} \arrow[d, "s_K"] \\ N \arrow[r, "v"'] & K \end{tikzcd}$$

Then $h \cdot t$ makes $f \cdot v$ and $g \cdot v$ homotopic. □

Observation 3.3. The homotopy relation does not depend on a choice of a cylinder object. The reason is that, for two cylinder objects

$$\nabla : K + K \xrightarrow{c_K} \overline{K} \xrightarrow{s_K} K$$

and

$$\nabla : K + K \xrightarrow{\overline{c}_K} \overline{\overline{K}} \xrightarrow{\overline{s}_K} K\,,$$

we always have a diagonal t in the square

$$\begin{tikzcd} K+K \arrow[r, "c_K"] \arrow[d, "\overline{c}_K"'] & \overline{K} \arrow[d, "s_K"] \\ \overline{\overline{K}} \arrow[ur, "t"] \arrow[r, "\overline{s}_K"'] & K \end{tikzcd}$$

Let $\mathcal{K}$ be a Quillen model category (see [H]) and let $\mathcal{L}$ consist of cofibrations and $\mathcal{R}$ of trivial fibrations. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system and any f, g homotopic in our sense are left homotopic in the standard sense. But the converse does not hold.

Any weak factorization system $(\mathcal{L}, \mathcal{R})$ gives rise to a Quillen model category if we take all morphisms of $\mathcal{K}$ as weak equivalences ([AHRT] 3.7). Then any two morphisms $f, g : K \to L$ are left homotopic because we have a model category cylinder

$$\nabla : K + K \xrightarrow{\mathrm{id}} K + K \xrightarrow{\nabla} K$$

(because ∇ is a weak equivalence).

Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. We get the quotient category

$$Q : \mathcal{K} \to \mathcal{K}/\sim \ .$$

Following 3.2, $Q(f) = Q(g)$ iff f and g are in the transitive closure of the homotopy relation $\sim$, i.e., iff there are $f_1, \dots, f_n$ such that

$$f \sim f_1 \sim \cdots \sim f_n \sim g \,.$$

Lemma 3.4. *Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. Then $P(f) = P(g)$ for any morphisms $f \sim g$.*

Proof. We have

$$P(s_K \cdot c_K^1) = P(\nabla \cdot i_1) = P(\nabla \cdot i_2) = P(s_K \cdot c_K^2) \,.$$

Since $P(s_K)$ is an isomorphism, we have $P(c_K^1) = P(c_K^2)$. Thus $P(f) = P(g)$ for $f \sim g$. □

We therefore have a unique functor T such that

$$\begin{tikzcd} \mathcal{K} \arrow[rr, "P"] \arrow[dr, "Q"'] & & \mathcal{K}[\mathcal{R}^{-1}] \\ & \mathcal{K}/\sim \arrow[ur, "T"'] & \end{tikzcd}$$

commutes. In general, one cannot expect that T is an equivalence.

Example 3.5. Let $\mathcal{K}$ have finite coproducts and consider the factorization system $\big(\mathrm{Iso}(\mathcal{K}), \mathrm{Mor}(\mathcal{K})\big)$. Then a cylinder object for K is $K+K$ and thus any two parallel morphisms are homotopic. Hence Q is the posetal reflection of $\mathcal{K}$. On the other hand, P is (up to equivalence), the projection of $\mathcal{K}$ to the set of connected components of $\mathcal{K}$.

If $(\mathcal{L}, \mathcal{R})$ is a weak factorization system and $\sim$ the associated homotopy then a morphism $f : K \to L$ is called a *homotopy equivalence* if there is $g : L \to K$ with $g \cdot f \sim \mathrm{id}_K$ and $f \cdot g \sim \mathrm{id}_L$. Every homotopy equivalence is sent by Q and, following 3.4, by P as well, to an isomorphism. Since $\sim$ is not transitive, Q inverts more morphisms than just homotopy equivalences. In fact, $Q(f)$ is an isomorphism iff there is g such that both $(g \cdot f, \mathrm{id}_K)$ and $(f \cdot g, \mathrm{id}_L)$ are in the transitive closure of $\sim$.

The following gives an example of a homotopy relation that is not transitive and of homotopy equivalent objects that are not bisimilar.

Example 3.6. Let $\mathbf{Set}^{\mathcal{X}}$ be the category of multigraphs with loops, i.e., $\mathcal{X}$ is the category

$$\begin{tikzcd} E \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & V \arrow[l, "v"'] \end{tikzcd}$$

where $r_1 \cdot v = r_2 \cdot v = \mathrm{id}_V$. Here, E is the object of edges, V is the object of vertices, r_1 and r_2 yield the initial and the final vertex of an edge and v yields loops. Let $\mathcal{L} = \mathit{Mono}$ be the class of all monomorphisms and $\mathcal{R}$ consist of morphisms $g : K \to L$ such that

(a) g is surjective on vertices and
(b) if vertices $g(a)$ and $g(b)$ are joined by an edge in L then a and b are joined by an edge in K.

In fact, this is the weak factorization system $\big({}^{\square}(\mathcal{C}^{\square}), \mathcal{C}^{\square}\big)$ from 2.9 where $\mathcal{C}$ consists of the embedding of an empty multigraph into a vertex and of the embedding of two vertices not connected by an edge to the edge.

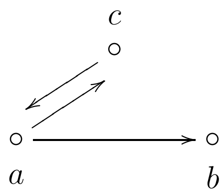
The cylinder object $c_K : K + K \to \overline{K}$ is obtained by joining the two copies $i_1(x)$ and $i_2(x)$ of a vertex x in K by two edges, one going from

$i_1(x)$ to $i_2(x)$ and the other going from $i_2(x)$ to $i_1(x)$. Moreover, having an edge from x to y, there is an edge going from $i_1(x)$ to $i_2(y)$ and an edge going from $i_2(x)$ to $i_1(y)$. This means that $\overline{K} = E' \times K$ where E' is the non-oriented edge, i.e., a complete graph on two vertices.

Morphisms $f, g : K \to L$ are homotopic iff for each vertex x of K, we have an edge from $f(x)$ to $g(x)$ and an edge from $g(x)$ to $f(x)$ in L. Moreover, having an edge from x to y, there is an edge going from $f(x)$ to $g(y)$ and an edge going from $g(x)$ to $f(y)$. Thus the homotopy relation is not transitive.

The multigraphs (loops are not depicted)

and



are homotopy equivalent. In fact, the first multigraph K is a retract of the second multigraph L via $u \cdot v = \mathrm{id}_K$ and the other composition $v \cdot u$ is homotopic to id_L. Hence K and L are isomorphic in $\mathcal{K}[\mathcal{R}^{-1}]$. But K and L are not bisimilar. Indeed, assume that there exist

$$K \xleftarrow{\ f\ } M \xrightarrow{\ g\ } L$$

with $f, g \in \mathcal{R}$. There are $x, y \in M$ with $g(x) = b$ and $g(y) = c$. Since $f(x)$ and $f(y)$ are joined by an edge in K, x and y are joined by an edge in M and thus b and c are connected by an edge in L; a contradiction.

Lemma 3.7. *Let $\mathcal{K}$ be a category having finite coproducts and $(\mathcal{L}, \mathcal{R})$ be a weak factorization system such that every morphism in $\mathcal{R}$ is a split epimorphism. Then every morphism in $\mathcal{R}$ is a homotopy equivalence.*

Proof. Let $r : K \to L$ be in $\mathcal{R}$ and consider f with $r \cdot f = \mathrm{id}_L$. It suffices to show that $f \cdot r \sim \mathrm{id}_K$. Consider the square

$$\begin{tikzcd}
K + K \arrow[r, "(f\cdot r, \mathrm{id}_K)"] \arrow[d, "c_K"'] & K \arrow[d, "r"] \\
\overline{K} \arrow[r, "r \cdot s_K"'] \arrow[ur, dashed, "t"] & L
\end{tikzcd}$$

which commutes because $r \cdot f \cdot r = r$. Since $c_K \in \mathcal{L}$ and $r \in \mathcal{R}$, there is a diagonal t, which is a homotopy from $f \cdot r$ to id_K. $\square$

Remark 3.8. We have proved a stronger statement: if $r \in \mathcal{R}$ is a split epimorphism then r is a homotopy equivalence. Following 2.8, if $0 \to L$ is in $\mathcal{L}$ then every $r : K \to L$ from $\mathcal{R}$ is a homotopy equivalence.

Theorem 3.9. *Let $\mathcal{K}$ be a category having finite coproducts and $(\mathcal{L}, \mathcal{R})$ be a weak factorization system such that every morphism in $\mathcal{R}$ is a split epimorphism. Then*

(1) *the categories $\mathcal{K}[\mathcal{R}^{-1}]$ and $\mathcal{K}/\sim$ are isomorphic,*
(2) *$\overline{\mathcal{R}} = \{f | Q(f)$ is an isomorphism$\}$ and*
(3) *$\mathcal{K}[\mathcal{R}^{-1}]$ is a locally small category.*

Proof. (1) Following 3.7, Q inverts all morphism from $\mathcal{R}$. Thus we get a unique functor U such that the triangle

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "P"] \arrow[dr, "Q"'] & & \mathcal{K}[\mathcal{R}^{-1}] \arrow[dl, "U"] \\
& \mathcal{K}/\sim &
\end{tikzcd}$$

commutes. Since both $U \cdot T$ and $T \cdot U$ are the identities, T is an isomorphism. It immediately yields (2) to (3). $\square$

Observation 3.10. Assume that $(\mathcal{L}, \mathcal{R})$ is a weak factorization system in a category $\mathcal{K}$ such that every morphism in $\mathcal{R}$ is a split epimorphism. Let $\mathcal{R}'$ consist of compositions $r \cdot f$ where $r \in \mathcal{R}$ and f splits some $s \in \mathcal{R}$, i.e., $s \cdot f = \mathrm{id}$. Then two objects K and L are bisimilar iff there is $h : K \to L$ in $\mathcal{R}'$.

(1) $\mathcal{R}'$ is closed under compositions.
Consider the composition

$$K_1 \xrightarrow{f_1} X_1 \xrightarrow{r_1} K_2 \xrightarrow{f_2} X_2 \xrightarrow{r_2} K_3$$

where $r_1, r_2 \in \mathcal{R}$ and $s_1 \cdot f_1 = \mathrm{id}_{K_1}$, $s_2 \cdot f_2 = \mathrm{id}_{K_2}$ for $s_1, s_2 \in \mathcal{R}$.
Let

$$\begin{tikzcd}
L \arrow[r, "\overline{s}_2"] \arrow[d, "\overline{r}_1"'] & X_1 \arrow[d, "r_1"] \\
X_2 \arrow[r, "s_2"'] & K_2
\end{tikzcd}$$

be a pulback. Then $\overline{r}_1, \overline{s}_2 \in \mathcal{R}$ and, since

$$s_2 \cdot f_2 \cdot r_1 = r_1 ,$$

there is a unique $t : X_1 \to L$ with

$$\overline{r}_1 \cdot t = f_2 \cdot r_1 \quad \text{and} \quad \overline{s}_2 \cdot t = \mathrm{id}_{X_1} \ .$$

Thus

$$r_2 \cdot f_2 \cdot r_1 \cdot f_1 = r_2 \cdot \overline{r}_1 \cdot t \cdot f_1$$

where $r_2 \cdot \overline{r}_1 \in \mathcal{R}$ and $s_1 \cdot \overline{s}_2 \cdot t \cdot f_1 = \mathrm{id}_{K_1}$, $s_1 \cdot \overline{s}_2 \in \mathcal{R}$. Hence $r_2 \cdot f_2 \cdot r_1 \cdot f_1 \in \mathcal{R}'$.

(2) If $g \cdot h$, $g \in \mathcal{R}'$ then $h \in \mathcal{R}'$. Let $g = r_1 \cdot f_1$ and $g \cdot h = r_2 \cdot f_2$ where $r_1, r_2 \in \mathcal{R}$ and f_i splits some $s_i \in \mathcal{R}$ for $i = 1, 2$.

Consider

$$\begin{tikzcd}
K_1 \arrow[r, "h"] \arrow[drr, "t"] \arrow[ddrr, "f_2"'] & K_2 \arrow[r, "f_1"] & X \arrow[r, "r_1"] & K_3 \\
 & & L \arrow[u, "\overline{r}_2"'] \arrow[d, "\overline{r}_1"'] & \\
 & & Y \arrow[uur, "r_2"'] &
\end{tikzcd}$$

where $r_1, r_2 \in \mathcal{R}$ and $s_1 \cdot f_1 = \mathrm{id}_{K_2}$, $s_2 \cdot f_2 = \mathrm{id}_{K_1}$ for $s_1, s_2 \in \mathcal{R}$. Take a pullback of r_1 and r_2 and consider the induced morphism t. Since $s_2 \cdot \overline{r}_1$, $s_1 \cdot \overline{r}_2 \in \mathcal{R}$,

$$(s_2 \cdot \overline{r}_1) \cdot t = s_2 \cdot f_2 = \mathrm{id}_{K_1} \,,$$

and

$$(s_1 \cdot \overline{r}_2) \cdot t = s_1 \cdot f_1 \cdot h = h \,,$$

we get $h \in \mathcal{R}'$.

(3) Any $g \in \mathcal{R}' \cap \mathcal{L}$ is a split monomorphism.

Let $g = r \cdot f$ where $s \cdot f = \mathrm{id}$ and $r, s \in \mathcal{R}$. Consider the square

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "r \cdot f"'] & X \arrow[d, "r"] \\
L \arrow[r, "\mathrm{id}_L"'] \arrow[ur, dashed, "t"] & L
\end{tikzcd}$$

If $g \in \mathcal{L}$ we get a diagonal t and we have

$$s \cdot t \cdot (r \cdot f) = s \cdot f = \mathrm{id}_K \ .$$

(4) $\mathcal{R}'$ does not need to have the 2-out-of-3 property.

Let $\mathbf{Set}^{\mathcal{X}}$ be the category of multigraphs with loops from 3.6 and consider the following multigraphs K and L (loops are not depicted):

$$\begin{tikzcd} \underset{a}{\circ} \arrow[r, shift left] & \underset{b}{\circ} \arrow[l, shift left] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & \overset{c}{\circ} \arrow[dl, shift right] \arrow[dr] & \\ \underset{a}{\circ} \arrow[ur, shift right] \arrow[rr, shift left] & & \underset{b}{\circ} \arrow[ll, shift left] \end{tikzcd}$$

Let $f : K \to L$ be the embedding (i.e., $f(a) = a$ and $f(b) = b$) and $s, t : L \to K$ split f by means of $s(c) = a$ and $t(c) = b$. Then $s \in \mathcal{R}$ and thus $f \in \mathcal{R}'$. Assume that $t \in \mathcal{R}'$. Then $t = r \cdot g$ where $r \in \mathcal{R}$ and $s' \cdot g = \mathrm{id}$ for some $s' \in \mathcal{R}$. Then there is an edge from $g(b)$ to $g(c)$ and thus and edge from b to c; a contradiction.

4. Stable equivalences

A full subcategory $\mathcal{M}$ of a category $\mathcal{K}$ is *weakly reflective* if for every object K in $\mathcal{K}$ there is a morphism $r_K : K \to K^*$ with $K^* \in \mathcal{M}$ such that every morphism $f : K \to M$, $M \in \mathcal{M}$ factorizes through r_K, i.e., $f = g \cdot r_K$.

Observation 4.1. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$ with finite products and $\mathcal{L}^{\triangle}$ the full subcategory of $\mathcal{K}$ consisting of objects M injective w.r.t. any morphism from $\mathcal{L}$. This means that for every $f : X \to Y$ in $\mathcal{L}$ and every $h : X \to M$ there is $g : Y \to M$ with $g \cdot f = h$. Then $\mathcal{L}^{\triangle}$ is weakly reflective in $\mathcal{K}$ where a weak reflection is given by a factorization

$$K \xrightarrow{\ r_K\ } K^* \xrightarrow{\ t\ } 1$$

with $r_K \in \mathcal{L}$ and $t \in \mathcal{R}$ of the unique morphism into the terminal object 1. Let $(\mathcal{L}^{\triangle})_{inj}$ consist of all morphism f such that each $M \in \mathcal{L}^{\triangle}$ is injective to f. Then

$$(\mathcal{L}^{\triangle})_{inj} = \{f \mid \text{there exists } g \in \mathcal{L} \text{ with } g = h \cdot f \text{ for some } h\}$$

which means that $(\mathcal{L}^{\triangle})_{inj}$ is the left cancellable closure of $\mathcal{L}$.

Indeed, it is easy to see that the left cancellable closure of $\mathcal{L}$ is contained in $\mathcal{L}^{\triangle}_{inj}$. Conversely, having $f : K \to L$ in $\mathcal{L}$, then a weak reflection $r_K : K \to K^*$ factorizes through f. Thus f belongs to the left cancellable closure of $\mathcal{L}$. Consequently, $\mathcal{L} = (\mathcal{L}^{\triangle})_{inj}$ whenever $\mathcal{L}$ is left cancellable.

Conversely, consider a weakly reflective full subcategory $\mathcal{M}$ of $\mathcal{K}$. Then, following [AHRT] 1.5, $\big(\mathcal{M}_{inj}, (\mathcal{M}_{inj})^{\square}\big)$ is a weak factorization system in $\mathcal{K}$. A weak factorization of $h : A \to B$ is

$$h : K \xrightarrow{\langle r_K, h\rangle} K^* \times L \xrightarrow{p_2} L\,.$$

Therefore, weak factorization systems $(\mathcal{L}, \mathcal{R})$ in $\mathcal{K}$ with $\mathcal{L}$ left cancellable are precisely $\big(\mathcal{M}_{inj}, (\mathcal{M}_{inj})^{\square}\big)$ for a weakly reflective full subcategory $\mathcal{M}$ of $\mathcal{K}$.

Let $\mathcal{M}$ be a weakly reflective full subcategory of an additive category $\mathcal{K}$. Morphisms $f, g : K \to L$ are called $\mathcal{M}$-*stably equivalent* if $f - g$ factorizes through an object M from $\mathcal{M}$. It is easy to see that this is an equivalence relation compatible with the composition. One gets the stable category $\mathcal{K}/\mathcal{M}$ and the projection $S : \mathcal{K} \to \mathcal{K}/\mathcal{M}$. We have the following results which are almost completely contained in [B] 4.5.

Lemma 4.2. *Let $\mathcal{M}$ be a weakly reflective full subcategory of an additive category $\mathcal{K}$. Then morphisms $f, g : K \to L$ are $\mathcal{M}$-stably equivalent iff $f \sim g$ with respect to the weak factorization system $\big(\mathcal{M}_{inj}, (\mathcal{M}_{inj})^{\square}\big)$.*

Proof. Following 4.1, cylinder objects are

$$K \oplus K \xrightarrow{c_K} K^* \oplus K^* \oplus K \xrightarrow{p_3} K$$

where $p_i \cdot c_K^j = r_K$ for $i = 1, 2$ and $j = 1, 2$. Of course, $p_3 \cdot c_K^j = \mathrm{id}_K$ for $j = 1, 2$. Hence $c_K^1 - c_K^2$ factorizes through $K^* \oplus K^*$ and thus c_K^1 and c_K^2 are $\mathcal{M}$-stably equivalent. Since $\mathcal{M}$-stable equivalence is compatible with the composition, $f \sim g$ implies that f and g are $\mathcal{M}$-stably equivalent. Conversely, let f and g be $\mathcal{M}$-stably equivalent, i.e., we have

$$f - g : K \xrightarrow{u} M \xrightarrow{v} L$$

with $M \in \mathcal{M}$. Then $f \sim g$ via

$$h : K^* \oplus K^* \oplus K \longrightarrow L$$

given by $h \cdot i_1 = 0$, $h \cdot i_2 = -v \cdot t$ and $h \cdot i_3 = f$ where

$$\begin{tikzcd} K \arrow[rr, "u"] \arrow[dr, "r_K"'] & & M \\ & K^* \arrow[ur, "t"'] & \end{tikzcd}$$

$\square$

The proof contains the description of a cylinder object using weak reflections to $\mathcal{M}$, which may be useful in homotopy theory of additive categories. We will need this in the following proof.

Consequently, homotopy equivalences coincide with $\mathcal{M}$-stable equivalences, i.e., with morphisms f admitting g with $g \cdot f$ and $f \cdot g$ $\mathcal{M}$-stably equivalent to the identities. A monomorphism f is called an $\mathcal{M}$-*monomorphism* if its cokernel belongs to $\mathcal{M}$ and an epimorphism g is called an $\mathcal{M}$-*epimorphism* if its kernel belongs to $\mathcal{M}$.

Theorem 4.3. *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in an additive category $\mathcal{K}$ such that $\mathcal{L}$ is left cancellable. Then the following conditions are equivalent for a morphism h:*

(i) *$h \in \overline{\mathcal{R}}$,*
(ii) *h is a homotopy equivalence,*
(iii) *h is an $\mathcal{L}^{\triangle}$-stable equivalence, and*
(iv) *$h = g \cdot f$ with f a split $\mathcal{L}^{\triangle}$-monomorphism and g a split $\mathcal{L}^{\triangle}$-epimorphism.*

Proof. (i) $\Rightarrow$ (iii). Consider $h \in \mathcal{R}$ and the factorization

$$h : K \xrightarrow{\langle r_K, h\rangle} K^* \oplus L \xrightarrow{p_2} L$$

from 4.1. We have

$$\begin{tikzcd} K \arrow[r, "\mathrm{id}_K"] \arrow[d, "\langle r_K, h\rangle"'] & K \arrow[d, "h"] \\ K^* \oplus L \arrow[r, "p_2"'] \arrow[ur, dashed, "t"] & L \end{tikzcd}$$

and thus h is a retract of p_2 in $\mathcal{K}^{\to}$ via

$$\begin{tikzcd} K \arrow[r, "h"] \arrow[d, "\langle r_K, h\rangle"'] & L \arrow[d, "\mathrm{id}_L"] \\ K^* \oplus L \arrow[r, "p_2"'] & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K \arrow[r, "h"] & L \\ K^* \oplus L \arrow[r, "p_2"'] \arrow[u, "t"] & L \arrow[u, "\mathrm{id}_L"'] \end{tikzcd}$$

Since $p_2 \cdot i_2 = \mathrm{id}_L$ and $i_2 \cdot p_2 = \mathrm{id}_{K^* \oplus L} - i_1 \cdot p_1$, p_2 is an $\mathcal{L}^{\triangle}$-stable equivalence with an $\mathcal{L}^{\triangle}$-stable inverse i_2. Hence $S : \mathcal{K} \to \mathcal{K}/\mathcal{L}^{\triangle}$ factorizes through $P : \mathcal{K} \to \mathcal{K}[\mathcal{R}^{-1}]$ and thus each morphism from $\overline{\mathcal{R}}$ is an $\mathcal{L}^{\triangle}$-stable equivalence.
(ii) $\Leftrightarrow$ (iii) following 4.2
(iii) $\Rightarrow$ (iv). Let $h = g \cdot f$ be a factorization of an $\mathcal{L}^{\triangle}$-stable equivalence with $f \in \mathcal{L}$ and $g \in \mathcal{R}$. Following the proof of (i) $\Rightarrow$ (iii), g is a retract of a split $\mathcal{L}^{\triangle}$-epimorphisms p_2. Hence g is a split $\mathcal{L}^{\triangle}$-epimorphism.

Since (i) $\Rightarrow$ (iii), g is an $\mathcal{L}^{\triangle}$-stable equivalence and thus f is an $\mathcal{L}^{\triangle}$-stable equivalence. Let t be an $\mathcal{L}^{\triangle}$-stable inverse of f. Then $t \cdot f$ is $\mathcal{L}^{\triangle}$-stably equivalent to the identity and thus we have a factorization

$$\mathrm{id}_K - t \cdot f : K \xrightarrow{\ u\ } M \xrightarrow{\ v\ } K$$

through $M \in \mathcal{L}^{\triangle}$. Since $f \in \mathcal{L}$, we have a factorization

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "u"'] & & L \arrow[dl, "w"] \\ & M & \end{tikzcd}$$

We have

$$\mathrm{id}_K - t \cdot f = v \cdot u = v \cdot w \cdot f$$

and thus

$$\mathrm{id}_K = (v \cdot w + t) \cdot f\,.$$

Thus f is a split monomorphism. Consequently, f has a cokernel $g : L \to P$ which is a split epimorphism.

Consider a morphism h such that $S(h \cdot f) = 0$ in $\mathcal{K}/\mathcal{L}^{\triangle}$. Then $h \cdot f$ factorizes through $N \in \mathcal{L}^{\triangle}$

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "r"'] & L \arrow[d, "h"] \arrow[dl, dashed, "p"] \\ N \arrow[r, "s"'] & X \end{tikzcd}$$

Since $f \in \mathcal{L}$, there is $p : L \to N$ with $p \cdot f = r$. We have

$$(h - s \cdot p) \cdot f = h \cdot f - s \cdot p \cdot f = h \cdot f - s \cdot r = 0$$

and thus there is $q : P \to X$ such that

$$q \cdot g = h - s \cdot p\,.$$

Hence $S(q \cdot g) = S(h)$ and thus $S(g)$ is a weak cokernel of $S(f)$ in $\mathcal{K}/\mathcal{L}^{\triangle}$. Since $S(g)$ is a split epimorphism, $S(g)$ is a cokernel of $S(f)$ in $\mathcal{K}/\mathcal{L}^{\triangle}$. However, $S(f)$ is an isomorphism and therefore P is the null object in $\mathcal{K}/\mathcal{L}^{\triangle}$. Consequently, $P \in \mathcal{L}^{\triangle}$, which proves that f is a split $\mathcal{L}^{\triangle}$-monomorphism.

(iv) $\Rightarrow$ (i) Let $f : K \to L$ be a split $\mathcal{L}^{\triangle}$-monomorphism. Then f is an injection of a biproduct

$$f : K \longrightarrow K \oplus M$$

where $M \in \mathcal{L}^{\triangle}$. Since the corresponding projection belongs to $\mathcal{R}$ (see 4.1 or the proof of 1.6 in [AHRT]), we get $f \in \overline{\mathcal{R}}$. Any split $\mathcal{L}^{\triangle}$-epimorphism $g : K \to L$ is a projection of a biproduct

$$L \oplus M \longrightarrow L$$

with $M \in \mathcal{L}^{\triangle}$ and thus belongs to $\mathcal{R}$. $\square$

Observation 4.4. Following the proof of (iii) $\Rightarrow$ (iv), any $g \in \mathcal{R}$ is a split $\mathcal{L}^{\triangle}$-epimorphism. Conversely, following (iv) $\Rightarrow$ (i), any split $\mathcal{L}^{\triangle}$-epimorphism belongs to $\mathcal{R}$. Thus compositions $g \cdot f$ from (iv) are precisely morphism from $\mathcal{R}'$ (cf. 3.10). Consequently, homotopy equivalent objects are precisely bisimilar objects.

Example 4.5. Consider the category R-**Mod** of R-modules and the class $\mathcal{L} = \mathit{Mono}$ of all monomorphisms. Then $\mathcal{L}^{\triangle}$ consists of injective R-modules and $\mathcal{R} = \mathit{Mono}^{\square}$ of all epimorphisms with an injective kernel. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system, $\mathcal{L}$ is left cancellable and R-**Mod** $/\mathcal{L}^{\triangle}$ is the usual stable category of modules.

5. Preshaves over posets

An object K of a category $\mathcal{K}$ is called *indecomposable* if the hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves binary coproducts.

Proposition 5.1. *Let $\mathcal{K}$ be a category with finite coproducts and $(\mathcal{L}, \mathcal{R})$ a weak factorization system such that $\mathcal{R} = \mathcal{C}^{\square}$ where every morphism from $\mathcal{C}$ has an indecomposable domain. Then $\mathcal{K}/\sim$ is equivalent to a poset.*

Proof. If suffices to show that $f \sim g$ for each $f, g : K \to L$. Consider a commutative square

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dd, "f"'] \arrow[dr, dashed, "u'"'] & & K+K \arrow[dd, "\nabla"] \\
& K \arrow[ur, "i_1"'] \arrow[dr, "\mathrm{id}_K"] & \\
Y \arrow[ur, "v"'] \arrow[rr, "v"'] & & K
\end{tikzcd}$$

with $f \in \mathcal{C}$. Then X is indecomposable and thus u factorizes through one of the coproduct injections, say, $u = i_1 \cdot u'$. Then

$$i_1 \cdot v \cdot f = i_1 \cdot \nabla \cdot u = i_1 \cdot \nabla \cdot i_1 \cdot u' = i_1 \cdot u' = u$$

and $\nabla \cdot i_1 \cdot v = v$. Hence $\nabla \in \mathcal{R}$ and thus the cylinder object is $\overline{K} = K + K$ with

$$c_K = \mathrm{id}_{K+K} : K + K \longrightarrow K + K\,.$$

Consequently, $f \sim g$ for any $f, g : K \to L$. $\square$

Remark 5.2. If $\mathcal{K}/\sim$ is equivalent to a poset then two objects K and L have $QK \cong QL$ iff there are morphisms both $K \to L$ and $L \to K$.

Proposition 5.3. *Let $\mathcal{K}$ have finite limits and finite coproducts and $(\mathcal{L}, \mathcal{R})$ be a weak factorization system such that $\mathcal{K}/\sim$ is equivalent to a poset. Then $\mathcal{K}[\mathcal{R}^{-1}]$ is equivalent to a poset.*

Proof. Following 2.3, $\mathcal{K}[\mathcal{R}^{-1}]$ is a quotient of the category of spans

$$K \xleftarrow{\ r\ } X \xrightarrow{\ f\ } L$$

with $r \in \mathcal{R}$. Consider two spans from K to L

$$\begin{tikzcd}
K & X_1 \arrow[l, "r_1"'] \arrow[r, "f_1"] & L \\
X_2 \arrow[u, "r_2"] \arrow[d, "f_2"'] & X \arrow[l, "\overline{r}_1"] \arrow[u, "\overline{r}_2"'] & \\
L & &
\end{tikzcd}$$

and a pullback of r_1 and r_2. Following our assumption the objects QX and QL are either isomorphic in $\mathcal{K}/\sim$ or $Q(f_1 \cdot \overline{r}_2) = Q(f_2 \cdot \overline{r}_1)$. Following 3.4, the objects PX and PL are either isomorphic in $\mathcal{K}[\mathcal{R}^{-1}]$ or $P(f_1 \cdot \overline{r}_2) = P(f_2 \cdot \overline{r}_1)$. In the first case, $PK \cong PL$. In the second case, we have

$$\begin{aligned}
P(f_1) \cdot P(r_1)^{-1} &= P(f_1) \cdot P(\overline{r}_2) \cdot P(\overline{r}_2)^{-1} \cdot P(r_1)^{-1} \\
&= P(f_1) \cdot P(\overline{r}_2) \cdot P(\overline{r}_1)^{-1} \cdot P(r_2)^{-1} \\
&= P(f_2) \cdot P(\overline{r}_1) \cdot P(\overline{r}_1)^{-1} \cdot P(r_2)^{-1} = P(f_2) \cdot P(r_2)^{-1}\,.
\end{aligned}$$

Thus the starting spans yield the same morphism in $\mathcal{K}[\mathcal{R}^{-1}]$. $\square$

Remark 5.4. If all morphisms in $\mathcal{R}$ are split epimorphisms then 5.3 immediately follows from 3.9.

Observation 5.5. Let $\mathcal{P}$ be a poset and consider the category $\mathcal{K} = \mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$ of presheaves on $\mathcal{P}$. Let $\mathcal{P}_\perp$ be the full subcategory of $\mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$ consisting of the image of $\mathcal{P}$ in the Yoneda embedding $Y : \mathcal{P} \to \mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$ with the initial presheaf 0 added. Then $\mathcal{P}_\perp$ is nothing else than $\mathcal{P}$ with a new initial element added. Following 2.9, we get a weak factorization system $\big({}^\square(\mathcal{P}_\perp^\square), \mathcal{P}_\perp^\square\big)$ in $\mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$. Since all objects in $\mathcal{P}_\perp$ are indecomposable, it follows from 5.1 and 5.3 that both categories $\mathcal{K}/\sim$ and $\mathcal{K}[(\mathcal{P}_\perp^\square)^{-1}]$ are equivalent to posets. Bisimilarity in this situation was used in [JNW] to formalize bisimilarity of processes. $\mathcal{P}_\perp^\square$ is the class of $\mathcal{P}_\perp$-open maps. 5.1 and 3.9 show that inverting all $\mathcal{P}_\perp$-open maps gives the category that has the same objects as $\mathbf{Set}^{\mathcal{P}^{op}}$ and an arrow $K \to L$ iff there is some arrow (simulation) $K \to L$ in $\mathbf{Set}^{\mathcal{P}^{op}}$.

Proposition 5.6. *Let $\mathcal{P}$ be a poset. Then the following conditions are equivalent:*

(i) ${}^\square(\mathcal{P}_\perp^\square)$ *coincides with the class Mono of all monomorphisms,*
(ii) *for every element $x \in \mathcal{P}$, any non-empty subset of $\{y \in \mathcal{P} | y \leq x\}$ has a greatest element.*

Proof. (i)$\Rightarrow$ (ii): Assume (i) and consider $x \in \mathcal{P}$ and a non-empty subset Z of $\{y \in \mathcal{P} | y \leq x\}$. Let K be a subfunctor of the representable functor $Y(x) : \mathcal{P}^{\mathrm{op}} \to \mathbf{Set}$ given as follows

$$K(p) = \begin{cases} 1 & \text{if} \quad p \leq z \quad \text{for some } z \in Z \\ 0 & \text{otherwise} \end{cases}$$

(here $0 = \emptyset$ and $1 = \{\emptyset\}$). Then the embedding $K \to Y(x)$ belongs to ${}^\square(\mathcal{P}_\perp^\square)$ and thus, following 2.9, is a retract of a transfinite composition g_λ of a chain $g_{ij} : K_i \to K_j$, $i \leq j < \lambda$ (λ is a limit ordinal) where $K_0 = K$, g_{ii+1} is a pushout of a morphism in $\mathcal{P}_\perp$ and, for j limit, g_{ij} is a colimit of g_{ik}, $i \leq k < j$. Let i be the smallest ordinal $i \leq \lambda$ such that $g_i = g_{0i}$ factorizes through a representable functor. It makes sense because g_λ has this property. Since representable functors are finitely presentable (cf. [AR] 1.2.(7)), i is either an isolated ordinal or $i = 0$.

Assume that $i = j+1$. Then we have a pushout below where $A, B \in \mathcal{P}_\perp$ and a factorization of g_i through a representable functor C

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
K \arrow[r, "g_j"] \arrow[dr, "h"'] & K_j \arrow[r, "g_{ji}"] & K_i \\
 & C \arrow[ur, "t"'] &
\end{tikzcd}$$

Then either $t = g_{ji} \cdot p$ for some p or $t = v \cdot q$ for some q. In the first case,

$$g_{ji} \cdot g_j = g_i = t \cdot h = g_{ji} \cdot p \cdot h$$

and thus $p \cdot h = g_j$ because g_{ji} is a monomorphism. Hence g_j factorizes through a representable functor, which contradicts the definition of i. In the second case,

$$g_{ji} \cdot g_j = t \cdot h = v \cdot q \cdot h$$

and thus there is a unique $w : K \to A$ such that

$$u \cdot w = g_j \quad \text{and} \quad f \cdot w = q \cdot h\,.$$

Since $K \neq 0$, we have $A \neq 0$ as well and thus A is representable. Hence g_j factorizes through a representable functor again; a contradiction.

Therefore $i = 0$, which means that $g_0 = \mathrm{id}_K$ factorizes through a representable functor. Thus K is a retract of a representable functor and, since $\mathcal{P}$ is a poset, K is representable. Thus Z has the greatest element.

(ii) $\Rightarrow$ (i) The condition (ii) clearly means that subfunctors of representable functors are representable or 0. Since $\mathcal{P}$ is a poset, quotients of representable functors are representable. Following the proof of [Be] 1.12, any monomorphism in $\mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$ belongs to ${}^\square(\mathcal{P}_\perp^\square)$. $\square$

Corollary 5.7. *Let $\mathcal{P}$ be a poset such that, for every element $x \in \mathcal{P}$, any non-empty subset of $\{y \in \mathcal{P} | y \leq x\}$ has a greatest element. Then $\mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}[(Mono^\square)^{-1}]$ is equivalent to a poset.*

Example 5.8. Let $\mathcal{P}$ be a two-element chain. Then $\mathbf{Set}^{\mathcal{P}^{\mathrm{op}}} = \mathbf{Set}^{\to}$ is the category of maps and $\mathcal{P}_\perp$ has three elements $o_0 : 0 \to 0$, $o_1 : 0 \to 1$

and $\mathrm{id}_1 : 1 \to 1$. Then a morphism $(u_1, u_2) : f \to g$

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "u_1"'] & A_2 \arrow[d, "u_2"] \\ B_1 \arrow[r, "g"'] & B_2 \end{tikzcd}$$

belongs to $\mathcal{P}_\perp^\square = Mono^\square$ iff u_2 is surjective and for every $a \in A_2$ and $b \in B_1$ with $g(b) = u_2(a)$ there is $c \in A_1$ such that $f(c) = a$ and $u_1(c) = b$. This implies that u_1 is surjective as well. In other words, (u_1, u_2) is a (surjective) bisimulation, that is, using the transition system notation $b_0 \to b$ for $g(b) = b_0$ and $a \to c$ for $f(c) = a$, it holds that for any "transition" $b_0 \to b$ and any a with $u_2(a) = b_0$ there is c such that $a \to c$ and $u_1(c) = b$.

Following 5.7, we have that $\mathbf{Set}^\to[(Mono^\square)^{-1}] = \mathbf{Set}^\to/\sim$ is equivalent to a poset. The objects id_1 and $\mathrm{id}_1 + o_1$ are isomorphic in this category because there are morphisms

$$\mathrm{id}_1 \to \mathrm{id}_1 + o_1 \quad \text{and} \quad \mathrm{id}_1 + o_1 \to \mathrm{id}_1 \ .$$

But the objects id_1 and $\mathrm{id}_1 + o_1$ are not bisimilar. In fact, assume that there are $f : A \to B$ and morphisms

$$\mathrm{id}_1 \xleftarrow{(u_1,u_2)} f \xrightarrow{(v_1,v_2)} \mathrm{id}_1 + o_1$$

in $Mono^\square$. Then (u_1, u_2) makes f surjective and thus $\mathrm{id}_1 + o_1$ is surjective as well; a contradiction.

References

[AHRT] J. Adámek, H. Herrlich, J. Rosický and W. Tholen, *Weak factorization systems and topological functors*, Appl. Categ. Structures 10 (2002), 237–249.

[AHS] J. Adámek, H. Herrlich and G. Strecker, *Abstract and Concrete Categories*, Wiley 1990.

[AR] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, Cambrige Univ. Press 1994.

[Be] T. Beke, *Sheafifiable homotopy model categories*, Math. Proc. Cambridge Philos. Soc. 129 (2000), 447–475.

[B] A. Beligiannis, *Homotopy theory of modules and Gorenstein rings*, Math. Scand. 88 (2001), 1–41.

[Bor] F. Borceux, *Handbook of Categorical Algebra*, Vol. I, Cambridge Univ. Press 1995.

[Bo] A. K. Bousfield, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra 9 (1977), 207–220.

[GZ] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag, Berlin 1967.

[H] P. S. Hirschhorn, *Model Categories and their Localizations*, American Mathematical Society 2003.

[JNW] A. Joyal, M. Nielsen and G. Winskel, *Bisimulation from open maps*, Inform. and Comput. 127 (1996), 164–185.

[KP] K. H. Kamps and T. Porter, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific, Singapore 1997.

[Q] D. Quillen, *Homotopical Algebra*, Lecture Notes in Math. 43, Springer-Verlag, Berlin 1967.

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