On a Generalized Small-Object Argument for the Injective Subcategory Problem

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Abstract

For locally ranked categories $\mathcal{A}$, which include all locally presentable categories and the category **Top** of topological spaces, we prove that, given any set $\mathcal{H}$ of morphisms of $\mathcal{A}$, the full subcategory of all $\mathcal{H}$-injective objects is weakly reflective in $\mathcal{A}$. Moreover, weak reflections can be chosen to be $\mathcal{H}$-cellular morphisms, i.e., to be transfinite compositions of pushouts of $\mathcal{H}$-morphisms. This generalizes the well-known Small-Object Argument used in algebraic homotopy theory.

I. Introduction

One of the classical questions of category theory is the "orthogonal subcategory problem": given a class $\mathcal{H}$ of morphisms of a category $\mathcal{A}$, when is the full subcategory $\mathcal{H}^{\perp}$ of all objects orthogonal to $\mathcal{H}$ reflective? In algebraic homotopy theory an equally important problem is the following "injective subcategory problem". Given a class $\mathcal{H}$ of morphisms in $\mathcal{A}$, we can form the full subcategory

$$\mathcal{H}\text{-}\mathit{Inj}$$

of all objects A injective w.r.t. $\mathcal{H}$ (i.e., such that $\hom(A,-) : \mathcal{A} \to \mathbf{Set}$ maps members of $\mathcal{H}$ to epimorphisms). Typically, this subcategory is not

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reflective, but we can ask about the existence of weak reflections (defined like reflections, except that factorizations are not required to be unique). Furthermore, if weak reflections exist, they are not unique, and we may ask about the "quality" of the weak reflection morphisms. Homotopy theorists need weak reflections to be $\mathcal{H}$-*cellular*, i.e., to belong to the closure of $\mathcal{H}$ under isomorphisms, pushouts and transfinite composition (or, equivalently, under multiple pushouts). The "injective subcategory problem", then, is the question of whether, for a given class $\mathcal{H}$ of morphisms of $\mathcal{A}$, every object of $\mathcal{A}$ has an $\mathcal{H}$-cellular weak reflection in $\mathcal{H}$-*Inj*.

We present a solution which generalizes the well-known Small Object Argument already present in Gabriel-Zisman [GZ], Chapter VI, Proposition 5.5.1. In that argument, an object A is called *small* w.r.t. $\mathcal{H}$ provided that there exists an infinite cardinal λ such that $\hom(A, -)$ preserves colimits of λ-chains of $\mathcal{H}$-cellular morphisms. Suppose that $\mathcal{A}$ is cocomplete and $\mathcal{H}$ is a set of morphisms such that every object is small w.r.t. A; then the injective subcategory problem has an affirmative answer. Thus, in locally presentable categories we conclude that the answer is affirmative for all sets of morphisms. But what about such basic categories as the category **Top** of topological spaces? We introduce here the concept of *locally ranked category*: it is a cocomplete and cowellpowered category such that every object A has rank, i.e. there exists an infinite cardinal λ such that $\hom(A, -)$ preserves unions of λ-chains of strong momomorphisms. Since λ-presentable objects have that property, we have

$$\text{locally presentable} \Rightarrow \text{locally ranked.}$$

But also **Top** and other topological categories are locally ranked. Our main result is the following

Generalized Small-Object Argument. For every set of morphisms in a locally ranked category, the injective subcategory problem has an affirmative answer.

What about the orthogonal subcategory problem in locally ranked categories? G. M. Kelly proved [K, Theorem 10.2] that the answer is affirmative not only for sets of morphisms, but also for those proper classes such that all members but a subset are epimorphisms. It is not known to us whether our result above holds for such classes too. However, we do know that for each such class $\mathcal{H}$ the subcategory $\mathcal{H}$-*Inj* is *almost reflective*, i.e., every object has a weak reflection into $\mathcal{H}$-*Inj*, and $\mathcal{H}$-*Inj* is closed under retracts. Moreover, $\mathcal{H}$-*Inj* is *naturally almost reflective* in the sence of [AHRT], i.e., there exists an endofunctor $R : \mathcal{A} \to \mathcal{A}$ together with a natural transformation $\varrho : Id \to R$ such that for every object K of $\mathcal{A}$

$$(*) \qquad \varrho_K : K \to RK \text{ is a weak reflection of } K \text{ in } \mathcal{H}\text{-}Inj.$$

What we do not know is whether one can choose the weak reflection to be $\mathcal{H}$-cellular.

Our proof, for sets $\mathcal{H}$ of morphisms in locally ranked categories uses substantionally a technique developed in the disseration of J. Reiterman [R] and published in [KR]. A weak reflection of an object K into $\mathcal{H}$-*Inj* is constructed iteratively; the first step of iteration is presented by a *pointed endofunctor*, i.e., an endofunctor $C : \mathcal{A} \to \mathcal{A}$ together with a natural transformation $\eta : Id \to C$. That is, we start from $\eta_K : K \to C(K)$ and then iterate:

$$(1) \qquad K \xrightarrow{\eta_K} C(K) \xrightarrow{\eta_{C(K)}} C^2(K) \xrightarrow{\eta_{C^2(K)}} \cdots$$

using chain-colimits on limit steps. We prove that there exists an ordinal i such that $C^i(K)$ is $\mathcal{H}$-injective, and then $K \to C^i(K)$ is an $\mathcal{H}$-cellular weak reflection. However, the pointed functor we use does not in general satisfy the equation $\eta C = C\eta$. Hence, there is an alternative "obvious" iteration, viz.,

$$(2) \qquad K \xrightarrow{\eta_K} C(K) \xrightarrow{C\eta_K} C^2(K) \xrightarrow{C^2\eta_K} \cdots$$

This leads to natural weak reflections (not only for sets $\mathcal{H}$, but also for the above-mentioned classes); however, these weak reflections are probably not $\mathcal{H}$-cellular. This brings us to the following:

Open Problem. Given a set $\mathcal{H}$ of morphisms of a locally ranked category, does there exist a pointed endofunctor $\varrho : Id \to R$ satisfying $(*)$ and such that each ϱ_K is $\mathcal{H}$-cellular?

In the last section we apply the Generalized Small-Object Argument to extend the theorem on weak factorization systems constructed from a given set of morphisms from locally presentable categories to locally ranked ones. This has been formulated earlier for locally presentable categories by T. Beke, see [B].

II. Generalized Small-Object Argument

II.1 Convention. All categories throughout our paper are supposed to be locally small, i.e., hom-sets are (small) sets.

II.2 Definition. Given a class $\mathcal{H}$ of morphisms of a cocomplete category $\mathcal{A}$, we denote by

$$\mathrm{cell}\,(\mathcal{H})$$

the least class of morphisms containing $\mathcal{H}$ and all identity morphisms which is pushout-stable and closed under transfinite composition, i.e., contains the colimit cocones of all chains in $\mathrm{cell}\,(\mathcal{H})$. Members of $\mathrm{cell}\,(\mathcal{H})$ are called *$\mathcal{H}$-cellular morphisms.*

Remark. Pushout stability means, of course, that in any pushout, opposite to an $\mathcal{H}$-cellular morphism is always an $\mathcal{H}$-cellular morphism. Observe that closedness under transfinite composition can be substituted by closedness under (i) composition and (ii) multiple pushouts (meaning that for every small discrete cone of $\mathcal{H}$-cellular morphism the colimit cocone is formed by $\mathcal{H}$-cellular morphisms).

Example. (1) Let $\mathcal{A} = \mathbf{Set}$ and let $\mathcal{H}$ consist of the single morphism $\emptyset \to 1$. Then $\mathrm{cell}\,(\mathcal{H}) =$ monomorphisms. In fact, by pushout stabiltity we get that the inclusion $A \to A+1$ is in $\mathrm{cell}\,(\mathcal{H})$ for every set A, and transfinite composites then yield all monomorphisms $A \to B$.

(2) Let $\mathcal{A} = \mathbf{Top}$, the category of topological spaces and continuous maps, and let $\mathcal{H}$ consist of the single embedding

$$e : \{0, 1\} \hookrightarrow [0, 1]$$

of the discrete two-element space into the unit real interval. Then $\mathrm{cell}\,(\mathcal{H})$ consists of the extensions of topological spaces A obtained by iteratively glueing new paths to pairs of elements of A.

II.3 Subcategories $\mathcal{H}$-*Inj*. Given a class $\mathcal{H}$ of morphisms of $\mathcal{A}$, an object A is said to be *$\mathcal{H}$-injective* provided that for every member $h : H \to H'$ of $\mathcal{H}$ and every morphism $f : H \to A$ there exists $f' : H' \to A$ with $f = f' \cdot h$. We denote by

$$\mathcal{H}\text{-}\mathit{Inj}$$

the full subcategory of all $\mathcal{H}$-injective objects of $\mathcal{A}$.

Example: $\mathcal{A} = \mathbf{Set}$, $\mathcal{H} = \{0 \to 1\}$, then $\mathcal{H}$-*Inj* is the full subcategory of all nonempty sets. If $\mathcal{A} = \mathbf{Top}$ and $\mathcal{H} = \{e\}$ above, then $\mathcal{H}$-*Inj* is the full subcategory of all pathwise connected topological spaces.

II.4 Weak Reflections Constructed by Iteration. Given an object K of $\mathcal{A}$ we are going to construct a transfinite chain

$$K = K_0 \to K_1 \to \cdots \to K_i \to \cdots \to K_j \to \cdots \qquad (i \leq j \text{ in } Ord)$$

which will be proved to approximate a weak reflection of K in $\mathcal{H}$-*Inj* in the sense that, for every ordinal i,

$$\text{if } K_i \text{ is } \mathcal{H}\text{-injective, then } K_0 \to K_i \text{ is a weak reflection of } K;$$

Moreover, all members of that transfinite chain are $\mathcal{H}$-cellular morphisms.

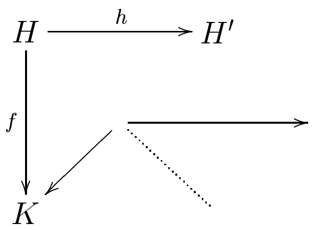
The first step of our construction, $K = K_0 \to K_1$, will be performed by a *pointed endofunctor* of $\mathcal{A}$, i.e., by an endofunctor together with a natural transformation from $Id_{\mathcal{A}}$. The subsequent steps will consist of iterating that endofunctor.

Notation. Let $\mathcal{H}$ be a set of morphisms in a cocomplete category $\mathcal{A}$. We define a pointed endofunctor

$$\eta : Id_{\mathcal{A}} \to C$$

of $\mathcal{A}$ as follows.

For every object K of $\mathcal{A}$ form a colimit of the following small diagram



consisting of all spans (f, h) where $h : H \to H'$ is any member of H and $f \in \hom(H, K)$. We denote a colimit cocone of that diagram as follows:

$$\begin{tikzcd} H \arrow[r, "h"] \arrow[d, "f"'] & H' \arrow[d, "f'"] \\ K \arrow[r, "\eta_K"'] & C(K) \end{tikzcd}$$

(Since f' depends on f *and* h, this is a slightly imprecise notation.)

To define C on morphisms $u : K \to \bar{K}$ of $\mathcal{A}$, observe that there exists a unique morphism

$$C(u) : C(K) \to C(\bar{K})$$

for which the diagrams

$$\begin{tikzcd}
H \arrow[r, "h"] \arrow[d, "f"] & H' \arrow[d, "f'"'] \arrow[dd, bend left=60, "(uf)'"] \\
K \arrow[r, "\eta_K"'] \arrow[d, "u"] & C(K) \arrow[d, "C(u)"'] \\
\bar{K} \arrow[r, "\eta_{\bar{K}}"'] & C(\bar{K})
\end{tikzcd}$$

commute for all (f, h) in the diagram above.

II.5 Example. Let $\mathcal{A} = \mathbf{Set}$ and $\mathcal{H} = \{0 \to 1\}$. Then $C(K) = K + 1$; more precisely, C is a coproduct of $Id_{\mathbf{Set}}$ and the constant functor with value 1.

II.6 Remark. As announced above, we are going to construct a weak reflection of an object K in $\mathcal{H}$-*Inj* by iterating the above pointed endofunctor C. There are, however, two "natural" possibilities of iteration : either

$$K \xrightarrow{\eta_K} C(K) \xrightarrow{\eta_{C(K)}} C^2(K) \xrightarrow{\eta_{C^2(K)}} \cdots$$

or

$$K \xrightarrow{\eta_K} C(K) \xrightarrow{C\eta_K} C^2(K) \xrightarrow{C^2\eta_K} \cdots$$

It turns out that each of them has its advantages: the first one leads to an $\mathcal{H}$-cellular weak reflection, the latter to a natural one (see II.17 below).

This makes a fundamental difference between the present injective-subcategory problem and the orthogonal-subcategory problem: in the latter, the corresponding functor C is well-pointed, i.e., $C\eta = \eta C$ (see [K]). Here, this equation does not hold in general, e.g., in the preceding example we have $C\eta \neq \eta C$.

II.7 A Weak-Reflection Chain. Let $\mathcal{H}$ be a set of morphisms of a cocomplete category $\mathcal{A}$. For every object K we define a transfinite chain of objects K_i ($i \in Ord$) and $\mathcal{H}$-cellular morphisms $k_{ij} : K_i \to K_j$ ($i \in j$) by the following transfinite inductions:

First step: $K_0 = K$;

Isolated step: $K_{i+1} = C(K_i)$ and $k_{i,i+1} = \eta_{K_i}$

Limit step: $K_j = \mathit{colim}\, K_i$ for every limit ordinal j, with a colimit cocone k_{ij} $(i < j)$.

To prove that the chain consists of $\mathcal{H}$-cellular morphisms, it is obviously sufficient to verify that η_K is $\mathcal{H}$-cellular for every object K. In fact, suppose that we form a pushout of each span of the diagram defining $C(K)$:

$$\begin{tikzcd} H \arrow[r, "h"] \arrow[d, "f"'] & H' \arrow[d, "f^*"] \\ K \arrow[r, "h^*"'] & P_{h,f} \end{tikzcd}$$

Then $h \in \mathcal{H}$ implies $h^* \in \mathrm{cell}\,(\mathcal{H})$, and η_K is obtained as a multiple pushout of all the morphisms h^*, thus, $\eta_K \in \mathrm{cell}\,(\mathcal{H})$.

II.8 Lemma. *The above weak-reflection chain is an approximation of a weak reflection of K in $\mathcal{H}$-Inj in the following sense:*

(a) *every $\mathcal{H}$-injective object is $\{k_{0i}\}$-injective for all ordinals i,*
and

(b) *if, for some ordinal i, K_i is $\mathcal{H}$-injective, then $k_{0i} : K \to K_i$ is a weak reflection in $\mathcal{H}$-Inj.*

Remark. In (b), $\mathcal{H}$-injectivity of K_i is equivalent to the property that the i-th step of the iteration, η_{K_i}, be a split monomorphism.

Proof. (a) Given $A \in \mathcal{H}$-*Inj* and a morphism $u : K \to A$, we show that there is a cocone $u_i : K_i \to A$ of the weak reflection chain above with $u_0 = u$, by transfinite induction.

Isolated step:

$$\begin{tikzcd} H \arrow[r, "h"] \arrow[d, "f"'] & H' \arrow[d] \arrow[ddl, bend left, "\hat{f}"] \\ K_i \arrow[r, "\eta_{K_i}"'] \arrow[d, "u_i"'] & C(K_i) \arrow[dl, dashed, "u_i+1"] \\ A & \end{tikzcd}$$

Since A is $\mathcal{H}$-injective, for each span (f, h) of the diagram defining $C(K_i)$ there exists $\widehat{f} : H' \to A$ with $\widehat{f}h = u_i f$. These morphisms, together with

u_i, form a cocone of that diagram, and we denote by $u_{i+1} : C(K_i) \to A$ the unique morphism that factors this cocone. In particular, $u_i = u_{i+1} \cdot \eta_{K_i}$.

(b) follows from (a). $\square$

II.9 Remark. The following corollary, due to Quillen [Q], is known as Small Object Argument. An object K is called *small* w.r.t. $\mathcal{H}$ if there exists an infinite cardinal λ such that $\hom(K, -)$ preserves colimits of λ-chains of $\mathcal{H}$-cellular morphisms.

II.10 Small Object Argument. Let $\mathcal{H}$ be a set of morphisms of a cocomplete category $\mathcal{A}$ such that all objects are small w.r.t. $\mathcal{H}$. Then $\mathcal{H}$-*Inj* is almost reflective with $\mathcal{H}$-cellular weak reflections.

Proof. There exists an infinite cardinal λ such that $\hom(H, -)$ preserves colimits of λ-chains of $\mathcal{H}$-cellular morphisms for every domain H of morphism in $\mathcal{H}$. We show that for every object K of $\mathcal{A}$, the object K_λ is $\mathcal{H}$-injective, which, by II.8, proves that $k_{0\lambda} : K \to K_\lambda$ is an $\mathcal{H}$-cellular weak reflection.

$$\begin{tikzcd} & H \arrow[r, "h"] \arrow[dl, "f^*"'] \arrow[d, "f"] & H' \\ K_i \arrow[r, "k_{i\lambda}"'] & K_\lambda & \end{tikzcd}$$

For every span (f, h) with $f : H \to K_\lambda$ there exists a factorization $f = k_{i\lambda} \cdot f^*$ for some $i < \lambda$ (since $\hom(H, -)$ preserves $K_\lambda = \underset{i<\lambda}{colim} K_i$), and this proves the $\mathcal{H}$-injectivity of K_λ : the morphism f factors through h since $\eta_{K_i} \cdot f^* = (f^*)' \cdot h$, hence

$$\begin{aligned} f &= k_{i\lambda} \cdot f^* \\ &= k_{i+1\lambda} \cdot \eta_{K_i} \cdot f^* \\ &= k_{i+1\lambda} \cdot (f^*)' \cdot h\,. \end{aligned}$$

$\square$

II.11 Remark. It is our goal to present a result of the same type as the Small Object Argument where, however, smallness w.r.t. $\mathcal{H}$ would be substituted by a (reasonably weak) property of the category $\mathcal{A}$, independent of $\mathcal{H}$. We call this property “locally ranked” which is intended to be reminiscent of “locally presentable” as introduced by Gabriel and Ulmer. The only drawback of this general result is the requirement that $\mathcal{A}$ be cowellpowered (which was not needed for the Small Object Argument). However, since most "everyday" categories are cowellpowered, we do not think that this is a real obstacle.

Below, we use the fact that every cocomplete and cowellpowered category has (epi, strong mono) factorization of morphisms, see 14.21 and 14C(d) in [AHS]. Recall that a monomorphism $m : A \to B$ is called *strong* provided that it has the diagonal-fill-in property w.r.t. epimorphisms. That is, given a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "e"'] & A \arrow[d, "m"] \\ Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where e is an epimorphism, there exist a diagonal morphism $Y \to A$ rendering both triangles commutative.

The whole theory below could be performed in any cocomplete cowellpowered category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms. We give an indication of this in the Remark following II.14; but in order to simplify the statement below, here we stick to $\mathcal{E}$ = epi and $\mathcal{M}$ = strong mono, in order to simplify the statements below.

II.12 Definition. A category $\mathcal{A}$ is called *locally ranked* provided that it is cocomplete and cowellpowered, and for every object K there exists an infinite cardinal λ (called rank of K) such that $\hom(K, -)$ preserves unions of λ-chains of strong subobjects.

Remark. Explicitly, *K has rank λ* provided that for every λ-chain of strong monomorphisms, $(A_i)_{i<\lambda}$, with a colimit cocone $a_i : A_i \to A$ $(i < \lambda)$, every morphism $f : K \to A$ factors through a_i for some $i < \lambda$. Every λ-presentable (in fact, every λ-generated) object has this property, but not vice versa: recall that e.g. in **Top** only the discrete spaces are λ-generated.

II.13 Examples. (1) All locally presentable categories (see [GU] or [AR]) are locally ranked. E.g., all varieties of algebras, the category of posets, and the category of simplicial sets are locally ranked.

(2) **Top**, the category of topological spaces and continuous maps, is locally ranked (but not locally presentable). Here strong monomorphisms are precisely the subspace embeddings. Every topological space K of cardinality smaller than λ, where λ is an infinite cardinal, has rank λ. In fact, given $f : K \to A = \mathit{colim}\, A_i$, there exists i such that $f[K] \subseteq a_i[A_i]$; hence f factors as $f = a_i \cdot f'$ in **Set**, and the continuity of f implies that of $f' : K \to A_i$

because $a_i : A_i \to A$ is a subspace embedding.

(3) More generally, all monotopological categories over **Set**, see [AHS], are locally ranked. These include the category of uniform spaces, and the category of Hausdorff topological spaces.

II.14 Theorem (Generalized Small Object Argument)**.** *Given a set $\mathcal{H}$ of morphisms in a locally ranked category, then every object has an $\mathcal{H}$-cellular weak reflection into $\mathcal{H}$-Inj.*

Remark. Instead of locally ranked, which refers to (epi, strong mono)-factorizations, we can formulate and prove our result for categories with a proper factorization system $(\mathcal{E}, \mathcal{M})$; i.e., $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ is a class of monomorphisms both closed under composition and such that $\mathcal{A} = \mathcal{M}\mathcal{E}$(i.e., every morphism f of $\mathcal{A}$ factors as $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$) and $\mathcal{M}$-morphisms have the diagonal fill-in property w.r.t. $\mathcal{E}$-morphisms. Let us call a category $\mathcal{A}$ *locally ranked w.r.t. a proper factorization system* $(\mathcal{E}, \mathcal{M})$ provided that $\mathcal{A}$ is cocomplete, $\mathcal{E}$-cowellpowered, and every object A has an *$\mathcal{M}$-rank*, i.e., a cardinal λ such that $\hom(A, -)$ preserves unions of λ-chains of $\mathcal{M}$-monomorphisms.

The above theorem holds for all locally ranked categories w.r.t. a factorization system.

Proof. We prove our results in the general $(\mathcal{E}, \mathcal{M})$-setting of the Remark above. According to II.8 we only need to show that for every object $K \in \mathcal{A}$ there is $i \in Ord$ with $K_i \in \mathcal{H}$-*Inj*. Denote by

$$e_{ij} : K_i \to X_{ij} (\in \mathcal{E}) \text{ and } m_{ij} : X_{ij} \to K_j (\in \mathcal{M})$$

an $(\mathcal{E}, \mathcal{M})$-factorization of $k_{ij} : K_i \to K_j$ (where $X_{ii} = K_i$, $e_{ii} = m_{ii} = \mathrm{id}$) in $\mathcal{A}$. For each $i \in Ord$ we obtain a chain of $\mathcal{E}$-morphisms $e_{ijj'} : X_{ij} \to X_{ij'}$ ($j \leq j'$ in *Ord*) by using the diagonal fill-in:

$$\begin{tikzcd}
K_i \arrow[rr, "e_{ij'}"] \arrow[d, "e_{ij}"'] & & X_{ij'} \arrow[d, "m_{ij'}"] \\
X_{ij} \arrow[r, "m_{ij}"'] \arrow[urr, "e_{ijj'}"] & K_j \arrow[r, "k_{jj'}"'] & K_{j'}
\end{tikzcd}$$

We obtain a two-dimensional diagram as follows:

$$
\begin{tikzcd}
K_0 \arrow[r, "k_{01}"] \arrow[d, "e_{001}"'] & K_1 \arrow[r, "k_{12}"] \arrow[d, "e_{112}"'] & K_2 \arrow[r] \arrow[d, "e_{223}"'] & \cdots \\
X_{01} \arrow[ur] \arrow[d, "e_{012}"'] & X_{12} \arrow[ur] \arrow[d, "e_{123}"'] & X_{23} \arrow[d, "e_{234}"'] & \\
X_{02} \arrow[ur] \arrow[d, "e_{023}"'] & X_{13} \arrow[ur] \arrow[d, "e_{134}"'] & X_{24} \arrow[d, "e_{245}"'] & \\
X_{03} \arrow[ur] \arrow[d, "e_{034}"'] & X_{14} \arrow[ur] \arrow[d, "e_{145}"'] & X_{25} \arrow[d, "e_{256}"'] & \\
\vdots & \vdots & \vdots &
\end{tikzcd}
$$

where the diagonal morphisms are $\mathcal{M}$-morphisms obtained, again, from the diagonal fill-in (for all $i \le i' \le j$):

$$
\begin{tikzcd}
K_i \arrow[r, "k_{ii'}"] \arrow[d, "e_{ij}"'] & K_{i'} \arrow[r, "e_{i'j}"] & X_{i'j} \arrow[d, "m_{i'j}"] \\
X_{ij} \arrow[urr] \arrow[rr, "m_{ij}"'] & & K_j
\end{tikzcd}
$$

Since $\mathcal{K}$ is $\mathcal{E}$-cowellpowered, for every ordinal i the chain $e_{iij} : K_i \to X_{ij}$ $(j \ge i)$ of $\mathcal{E}$-quotients of K_i is strationary, i.e., there exists $i^* \ge i$ such that all the quotients e_{iij} with $j \ge i^*$ are equivalent. In other words,

$$k_{ji^*} m_{ii^*} \in \mathcal{M} \text{ for all } j \ge i^* \tag{1}$$

We choose, for each i, our ordinal i^* so that $(-)^* : \mathit{Ord} \to \mathit{Ord}$ is monotone and define the iteration of $*$ as the following function $\varphi : \mathit{Ord} \to \mathit{Ord}$:

$$
\begin{aligned}
\varphi(0) &= 0 \\
\varphi(i+1) &= \varphi(i)^* \\
\varphi(j) &= \bigvee_{i<j} \varphi(i) \quad \text{for limit ordinals } j\,.
\end{aligned}
$$

Let (L_i) be the following chain:

$$L_i = X_{\varphi(i)\varphi(i)^*}$$

and the connecting morphisms l_{ij} are given by the diagonal fill-in for all $i \leq j$:

(2)
$$\begin{tikzcd}
K_{\varphi(i)} \arrow[r, "e_{f(i)f(i)^*}"] \arrow[d, "k_{\varphi(i)\varphi(j)}"'] & X_{\varphi(i)\varphi(i)^*} \arrow[d, "m_{\varphi(i)\varphi(i)^*}"] \arrow[ddl, "l_{ij}"'] \\
K_{\varphi(j)} \arrow[d, "e_{\varphi(j)\varphi(j)^*}"'] & K_{\varphi(i^*)^*} \arrow[d, "k_{\varphi(i)\varphi(j)^*}"] \\
X_{\varphi(j)\varphi(j^*)^*} \arrow[r, "m_{\varphi(j)\varphi(j)^*}"'] & K_{\varphi(j)^*}
\end{tikzcd}$$

Due to (1) we conclude

(3)
$$l_{ij} \in \mathcal{M} \quad \text{for all} \quad i \leq j$$

and we have a natural transformation

(4)
$$d_i = e_{\varphi(i)\varphi(i)^*} : K_{\varphi(i)} \to L_i \text{ in } \mathcal{E} \quad (i \in \mathit{Ord}\,).$$

Let us show that for every limit ordinal j we have

$$\begin{aligned} K_{\varphi(j)} &= \underset{i<j}{colim} L_i \quad \text{with colimit cocone} \\ c_{ij} &= k_{\varphi(i)^*\varphi(j)} \cdot m_{\varphi(i)\varphi(i)^*} \qquad (i<j)\,. \end{aligned} \tag{5}$$

Given a cocone $h_i : L_i \to H$ $(i < j)$ the cocone $h_i d_i : K_{\varphi(i)} \to L$ has a unique factorization through $K_{\varphi(j)} = \underset{i<j}{colim} K_{\varphi(i)}$. Thus, there is a unique $h : K_{\varphi(j)} \to H$ with

(6)
$$h \cdot k_{\varphi(i)\varphi(j)} = h_i d_i \qquad (i<j)\,.$$

We conclude that

(7)
$$h \cdot c_{ij} = h_i \qquad (i<j)$$

because d_i is an epimorphism with

$$\begin{aligned}
h \cdot c_{ij} \cdot d_i &= h \cdot k_{\varphi(i)i^*\varphi(j)} \cdot m_{\varphi(i)\varphi(i)^*} \cdot d_i && \text{by (5)} \\
&= h \cdot k_{\varphi(i)^*\varphi(j)} \cdot m_{\varphi(i)\varphi(i)^*} \cdot e_{\varphi(i)\varphi(i)^*} && \text{by (4)} \\
&= h \cdot k_{\varphi(i)^*\varphi(j)} \cdot k_{\varphi(i)\varphi(i)^*} && \\
&= h \cdot k_{\varphi(i)\varphi(j)} && \\
&= h_i d_i && \text{by (6).}
\end{aligned}$$

And h is unique because (7) implies (6).

We are ready to prove that $K_{\varphi(\lambda)}$ is $\mathcal{M}$-injective, whenever all domains of $\mathcal{M}$-morphisms have $\mathcal{M}$-rank λ. In fact, given

$$m : A \to A' \text{ in } \mathcal{M} \text{ and } f : A \to K_{\varphi(\lambda)}$$

then since $\hom(A, -)$ preserves the colimit $K_{\varphi(\lambda)}$ of the $\mathcal{M}$-chain $(L_i)_{i<\lambda}$, there exists $i < \lambda$ such that f factors through $c_{if(\lambda)} : L_i \to K_{\varphi(\lambda)}$, say

(8) $$f = c_{if(\lambda)} \cdot \bar{f} \text{ for some } \bar{f} : A \to L_i = X_{\varphi(i)\varphi(i)^*}$$

For the morphism

(9) $$g = m_{\varphi(i)\varphi(i)^*} \cdot \bar{f} : A \to K_{f(i)^*}$$

we have $g' : A' \to K_{f(i)^*+1}$ with

(10) $$g'm = k_{f(i)^*, f(i)^*+1} \cdot g$$

and we conclude

$$\begin{tikzcd}
 & & & A \arrow[r, "m"] \arrow[d, "f"] \arrow[ll, "g"'] \arrow[ddlll, bend right, "\bar{f}"'] & A' \arrow[dll, dashed, "g'"] \\
K_{\varphi(i)} \arrow[d] & K_{\varphi(i)^*} \arrow[r] & K_{\varphi(i)^*+1} \arrow[r] & K_{\varphi(\lambda)} & \\
X_{\varphi(i)\varphi(i)^*} \arrow[ur] & & & &
\end{tikzcd}$$

$$\begin{aligned}
f &= c_{i\varphi(\lambda)} \bar{f} && \text{by (8)} \\
&= k_{\varphi(i)^*\varphi(\lambda)} \cdot m_{\varphi(i)\varphi(i)^*} \cdot \bar{f} && \text{by (5)} \\
&= k_{\varphi(i)^*\varphi(\lambda)} \cdot g && \text{by (9)} \\
&= k_{\varphi(i)^*+1,\varphi(\lambda)} k_{\varphi(i)^*,\varphi(i)^*+1} \cdot g && \\
&= (k_{\varphi(i)^*+1,\varphi(\lambda)} \cdot g') \cdot m && \text{by (10).}
\end{aligned}$$

This conclude the proof that $K_{\varphi(\lambda)}$ is $\mathcal{M}$-injective. $\square$

II.15 Remark. Following [AHRT], we call a subcategory $\mathcal{D}$ of a category $\mathcal{A}$ *naturally almost reflective* provided that (i) $\mathcal{A}$ has a pointed endofunctor $\varrho : Id_{\mathcal{A}} \to R$ such that for every object K of $\mathcal{A}$ we have:

(∗) $$\varrho : K \to RK \text{ is a weak reflection of } K \text{ in } \mathcal{B},$$

and that (ii) $\mathcal{B}$ is closed under retracts in $\mathcal{A}$ (which is automatic in case $\mathcal{B} = \mathcal{H}\text{-}Inj$). The above proof of Theorem II.14 does not yield naturality: the ordinal i for which $k_{0i} : K \to K_i$ is a weak reflection of K in $\mathcal{B} = \mathcal{H}\text{-}Inj$ depends on K. This is a drawback in comparison with the Small Object Argument (II.10): there we had one ordinal λ for all objects K. Thus, a weak-reflection endofunctor of $\mathcal{A}$ is obtained by λ iterations of $\eta : Id \to C$. That is, define a λ-chain in $\mathcal{A}^{\mathcal{A}}$ by the following transfinite induction:

First step : C_0

Isolated step : $C_{i+1} = C \cdot C_i$ and $c_{i+1,j+1} = Cc_{ij}$;

Limit step : $C_j = \underset{i<j}{colim} C_i$ for limit ordinals j.

Then

$$c_{0\lambda} : Id \to C_\lambda$$

is a pointed endofunctor satisfying $(*)$ for $\mathcal{B} = \mathcal{H}\text{-}Inj$. This has been proved in II.10.

II.16 Open Problem: Is $\mathcal{H}\text{-}Inj$ naturally and $\mathcal{H}$-cellularly almost reflective in every locally ranked category $\mathcal{A}$ (and for every set $\mathcal{H}$ of morphisms in $\mathcal{A}$)?

That is, does there exist a pointed endofunctor $\varrho : Id_{\mathcal{A}} \to R$ such that for every object K, $(*)$ holds for $\mathcal{B} = \mathcal{H}\text{-}Inj$, and ϱ_K is also $\mathcal{H}$-cellular?

Example: The answer is affirmative for all locally presentable categories $\mathcal{A}$.

II.17 Remark. The above problem is really vexing since, as kindly pointed to us by Steven Lack, $\mathcal{H}\text{-}Inj$ *is* naturally almost reflective in $\mathcal{A}$. In fact, an object A is $\mathcal{H}$-injective iff η_A is a split monomorphism (cf. Remark II.8). In the terminology of [K] this is equivalent to A carrying the structure of an algebra of the pointed endofunctor $T = (C, \eta)$ of II.4 above. Thus, we have a canonical forgetful functor V from the category $Alg(T)$ of all algebras over T into $\mathcal{H}\text{-}Inj$. If α denotes a joint rank of all the domains and codomains of morphisms in $\mathcal{H}$, then the functor C preserves colimits of α-chains of morphisms from $\mathcal{M}$ (the right-hand class of our factorization system $(\mathcal{E}, \mathcal{M})$): see Theorem 10.1 of [K] for the analogous argument (in fact, the only modification needed is to restrict the pushout (10.1) used in [K] to the left-hand summands only: the resulting pushout is the pointed functor (C, η) above). It then follows from Theorem 15.6 of [K] that the category $Alg(T)$ of all algebras is reflective in the category $T \downarrow \mathcal{A}$ of all arrows $TX \to Y$. The natural forgetful functor $U : T \downarrow \mathcal{A} \to \mathcal{A}$ assigning to each of the arrows the object X has a left adjoint (sending X to $id : TX \to TX$), therefore, the functor V, a domain-codomain restriction of U, also has a left adjoint, and this yields the desired functorial weak reflection.

III. Weak Factorization Systems

III.1. In this section we apply Theorem II.14 to generalize a result on weak factorization systems from the realm of locally presentable categories to all locally ranked categories.

We use the notation

$$f \,\Box\, g$$

for the statement that the morphism $f : A \to B$ has the diagonal fill-in property w.r.t. $g : C \to D$, i.e., for every commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

there exists a diagonal

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur] & D \end{tikzcd}$$

making both triangles commutative. For every class $\mathcal{H}$ of morphisms in $\mathcal{A}$ we denote by $\mathcal{H}^\Box$ and ${}^\Box\mathcal{H}$ the classes obtained by the Galois correspondence induced by $\Box$:

$$\mathcal{H}^\Box = \{g \in \operatorname{mor}\mathcal{A}; h \,\Box\, g \text{ for all } h \in \mathcal{H}\}$$

and

$${}^\Box\mathcal{H} = \{f \in \operatorname{mor}\mathcal{A}; f \,\Box\, h \text{ for all } h \in \mathcal{H}\}\,.$$

III.2 Remark. Observe that an object K is $\mathcal{H}$-injective iff the unique morphism $K \to 1$ lies in $\mathcal{H}^\Box$.

Conversely, a morphism $g : C \to D$ lies in $\mathcal{H}^\Box$ iff g as an object of the comma-category $\mathcal{A} \downarrow D$, g is $\bar{\mathcal{H}}$-injective for the set $\bar{\mathcal{H}}$ of all morphisms

$$\begin{tikzcd}
H \arrow[rr, "h"] \arrow[dr] & & H' \arrow[dl] \\
& D &
\end{tikzcd}$$

of $\mathcal{A} \downarrow D$ with $h \in \mathcal{H}$. (See 12.4.2 in [H].)

The following definition appears in [B].

III.3 Definition. By a *weak factorization system* in a category $\mathcal{A}$ is meant a pair of classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{A}$ such that

(i) $\mathcal{A} = \mathcal{R}\mathcal{L}$

(ii) $\mathcal{R} = \mathcal{L}^{\square}$

and

(iii) $\mathcal{L} = {}^{\square}\mathcal{R}$

That is, every morphism of $\mathcal{A}$ has a factorization as an $\mathcal{L}$-morphism followed by an $\mathcal{R}$-morphism, and $\mathcal{L}$ and $\mathcal{R}$ determine each other in the Galois correspondence induced by the diagonal fill-in relation $\square$ on $\operatorname{mor}\mathcal{A}$.

III.4 Remark. (1) The above conditions (i)–(iii) are easily seen to be equivalent to the following:

(i) $\mathcal{A} = \mathcal{R}\mathcal{L}$,

(ii*) $\mathcal{L} \square \mathcal{R}$, i.e., $l \square r$ for all $l \in \mathcal{L}$, $r \in \mathcal{L}$,

and

(iii*) both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts in the category $A^{\to}$.

Recall that $\mathcal{A}^{\to}$ has as objects all morphisms of $\mathcal{A}$, and as morphisms commutative squares. Thus, (iii*) says that in every commutative diagram of $\mathcal{A}$ of the following form

$$\begin{tikzcd}
{} \arrow[rr, "\mathrm{id}"] \arrow[dr, "j"] \arrow[ddd, "f"'] & & {} \arrow[ddd, "f"] \\
& {} \arrow[ur, "q"] \arrow[d, "g"] & \\
& {} \arrow[dr, "p"] & \\
{} \arrow[ur, "i"] \arrow[rr, "\mathrm{id}"'] & & {}
\end{tikzcd}$$

$g \in \mathcal{L}$ implies $f \in \mathcal{L}$ and $g \in \mathcal{R}$ implies $f \in \mathcal{R}$. Or, equivalently, that the following implications hold:

whenever $i \cdot f \in \mathcal{L}$ and $p \cdot i = \mathrm{id}$, then $f \in \mathcal{L}$

and

whenever $f \cdot q \in \mathcal{R}$ and $q \cdot j = \mathrm{id}$, then $f \in \mathcal{R}$.

(2) In a weak factorization system,

$$\mathcal{L} \cap \mathcal{R} = \mathrm{Iso}\,.$$

In fact, this follows from (ii) and (iii) only (see [Ri])

III.5 Examples. (1) Every factorization system in a category (which can be defined as above except that "diagonalization property" is substituted by "unique diagonalization property", see [CHK]) is a weak factorization system, see 14.6(3) in [AHS].

(2) Every Quillen model category, given by morphisms classes
$\mathcal{F}$ (fibrations),
$\mathcal{C}$ (cofibrations),
and
$\mathcal{W}$ (weak equivalences)
has two prominent weak factorization systems:
$(\mathcal{C}, \mathcal{F}_0)$ where $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{W}$ (trivial fibrations)
and
$(\mathcal{C}_0, \mathcal{F})$ where $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{W}$ (trivial cofibrations),
see [Q].

(3) In **Set**, by (1) we have a (weak) factorization system (Epi, Mono). Also (Mono, Epi) is a weak factorization system. In fact:

(i) Every morphism $f : A \to B$ factors as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook] & & B \\ & A + B \arrow[ur, "{[f,\mathrm{id}]}"'] & \end{tikzcd}$$

(ii) Epi $\subseteq \mathrm{Mono}^{\square}$ because epimorphisms split:

given a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, hook] & C \arrow[d, shift left, "g"] \\ B \arrow[r, "v"'] & D \arrow[u, dashed, shift left, "i"] \end{tikzcd}$$

with $G_i = \mathrm{id}$, extend u to B by using $v \cdot i$ on $B - A$. Further $\mathrm{Mono}^\square \subseteq \mathrm{Epi}$ which is trivial since $\emptyset \to 1$ is a monomorphism.

(iii) ${}^\square\mathrm{Epi} \subseteq \mathrm{Mono}$ since for every morphism $f : A \to B$ in ${}^\square\mathrm{Epi}$, f is a split monomorphism due to the following square

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$

III.6 Definition. Let $\mathcal{H}$ be a class of morphisms in a cocomplete category $\mathcal{A}$. A morphism $f : A \to B$ is called an $\mathcal{H}$-*cofibration* provided that, in the comma category $A \downarrow \mathcal{A}$, it is a retract, of an $\mathcal{H}$-cellular morphism; that is: there exist $f' : A \to B'$ in $\mathrm{cell}\,(\mathcal{H})$ and commutative triangles

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ B \arrow[rr, "i"] & & B' \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ B' \arrow[rr, "r"] & & B \end{tikzcd}$$

with $r \cdot i = \mathrm{id}$.

Remark. We denote by

$$\mathrm{cof}(\mathcal{H})$$

the class of all $\mathcal{H}$-cofibrations. It is easy to show that

$$\mathrm{cof}({}^\square\mathcal{H}) = {}^\square\mathcal{H}$$

for every $\mathcal{H}$, see [H], 8.2.5-9 and 12.2.16. The following theorem generalizes the result of T. Beke [B] from locally presentable categories to locally ranked ones:

III.7 Theorem. *Let $\mathcal{H}$ be a set of morphisms in a locally ranked category. Then $(\mathrm{cof}(\mathcal{H}), \mathcal{H}^\square)$ is a weak factorization system.*

Proof. Put $\mathcal{R} = \mathcal{H}^\square$ and $\mathcal{L} = {}^\square\mathcal{R}$. Then we first prove that $(\mathcal{L}, \mathcal{R})$ is a weak factorization system, and then that $\mathcal{L} = \mathrm{cof}(\mathcal{H})$.

(a) Every morphism $f : X \to Y$ of $\mathcal{A}$ has an $(\mathcal{L}, \mathcal{R})$-factorization: The comma-category $\mathcal{A} \downarrow Y$ is obviously cocomplete and $\mathcal{E}$-cowellpowered, where $\mathcal{E}$ consists of all morphisms of $\mathcal{A} \downarrow Y$ whose underlying morphism in $\mathcal{A}$ is an epimorphism. Moreover, $\mathcal{A} \downarrow Y$ has the proper factorization system $(\mathcal{E}, \mathcal{M})$

where $\mathcal{M}$ consists of all morphisms whose underlying morphism in $\mathcal{A}$ is a strong monomorphism. Finally, every object $b : B \to Y$ of $\mathcal{A} \downarrow Y$ has an $\mathcal{M}$-rank (see Remark II.14): if B has rank λ in $\mathcal{A}$ then (B, b) has $\mathcal{M}$-rank λ in $\mathcal{A} \downarrow Y$. Consequently, for every set of morphisms of $\mathcal{A} \downarrow Y$ weak cellular reflections exist by II.14. We apply this to the set $\bar{\mathcal{H}}$ of all morphisms whose underlying morphism in $\mathcal{A}$ lies in $\mathcal{H}$. Thus, the object $f : X \to Y$ given above has an $\bar{\mathcal{H}}$-cellular weak reflection

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "r"'] & Y \\ & X^* \arrow[u, "f^*"'] \end{tikzcd}$$

in $\bar{\mathcal{H}}$-*Inj*. But $\bar{\mathcal{H}}$-*Inj* $= \mathcal{H}^{\square} = \mathcal{R}$, see III.2, thus

$$f^* \in \mathcal{R}\,.$$

Also, $r \in \operatorname{cell}(\bar{\mathcal{H}})$, i.e., $r \in \operatorname{cell}(\mathcal{H})$, which implies $r \in \operatorname{cell}(\mathcal{L})$ because $\mathcal{H} \subseteq {}^{\square}(\mathcal{H}^{\square}) = \mathcal{L}$ and by Remark III.5 we conclude

$$r \in \mathcal{L}\,.$$

(b) $\mathcal{L} = {}^{\square}\mathcal{R}$ by definition.

(c) $\mathcal{R} = \mathcal{L}^{\square}$ by definition of $\mathcal{H}$ and (a) above.

(d) $\mathcal{L} = \operatorname{cof}(\mathcal{H})$. In fact, it is clear that $\operatorname{cof}(\mathcal{H})$ is contained in $\mathcal{L} = {}^{\square}(\mathcal{H}^{\square})$, see Remark III.5. Conversely, given $f : X \to Y$ in $\mathcal{L}$, consider the above factorization $f = f^* r$, where $r \in \operatorname{cell}(\mathcal{H})$. It is sufficient to show that f is a retract of r in $\mathcal{A} \downarrow Y$; in fact, we can use the diagonal fill-in property:

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "f"'] & X^* \arrow[d, "f^*"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\square$

References

[AHS] J. Adámek, H. Herrlich and G. E. Strecker: *Abstract and Concrete Categories*, John Wiley, New York 1990.

[AHRT] J. Adámek, H. Herrlich, J. Rosický and W. Tholen: *Injective hulls are not natural*, submitted.

[AR] J. Adámek and J. Rosický: *Locally presentable and accessible categories*, Cambridge University Press, 1994.

[B] T. Beke: *Sheafifiable homotopy model categories*, Math. Proc. Cambr. Phil. Soc. 129 (2000), 447-475.

[CHK] Cassidy, M. Hebert, G. M. Kelly: *Reflective subcategories, localizations and factorization systems*, J. Austral. Math. Soc. 38 (1985), 287–329.

[GU] P. Gabriel and F. Ulmer: *Lokal präsentiebare Kategorien*, Springer Lecture Notes Math. 221, Springer-Verlag Berlin, 1971.

[GZ] P. Gabriel and M. Zisman: *Calculus of fractions and homotopy theory*, Springer-Verlag Berlin, 1967.

[H] Hirschhorn: *Localizations of model categories*, preprint 1998, `http://www.math.mit.edu/psh`.

[He] H. Herrlich: *Almost reflective subcategories of* **Top**, Topology Appl. 49 (1993), 251–264.

[K] M. Kelly: *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Austral. Math. Soc. 22 (1980), 1–84.

[KR] V. Koubek and J. Reiterman: *Categorical constructions of free algebras, colimits and completions of partial algebras*, J. Pure Appl. Algebra 14 (1979), 195–231.

[Q] D. Quillen: *Homotopical algebra*, Lecture Notes Math. 43, Springer-Verlag, Berlin 1967.

[R] J. Reiterman:*Categorical algebraic constructions*, Charles University dissertation (in Czech), Prague 1976.

[Ri] C.M.Ringel: *Diagonalisierungspaare I.*, Math. Z. 117 (1970), 240–266.

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