On von Neumann varieties

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To Aurelio Carboni, on his sixtieth birthday.

Abstract

We generalize to an arbitrary variety the von Neumann axiom for a ring. We study its implications on the purity of monomorphisms and the flatness of algebras.

Introduction

First, let us fix the context and the notation of this paper and review some notions that we need (see [2]). Let us mention at once that through the paper "quotient" means always "regular quotient" (coequalizer of two morphisms) and "projective" means always "regular projective" (projective with respect to regular quotients). As usual, given a category $\mathcal{V}$ with filtered colimits, an object $M \in \mathcal{V}$ is finitely presentable when the representable functor $\mathcal{V}(M, -)$ preserves filtered colimits.

- $\mathcal{V}$ is a variety, that is, the category of models of a many sorted algebraic theory. A variety is always a locally finitely presentable, regular and exact category. Its objects are also referred to as "algebras".
- $\mathcal{P}$ is the full subcategory of finitely presentable projective objects in $\mathcal{V}$. This category $\mathcal{P}$ is Cauchy complete (i.e. every idempotent splits) and is closed in $\mathcal{V}$ under finite coproducts. The dual category of $\mathcal{P}$ is the so-called "canonical theory" of $\mathcal{V}$ (see [1]): $\mathcal{V}$ is equivalent to the category of finite product preserving functors $\mathcal{P}^{\mathsf{op}} \longrightarrow \mathsf{Set}$, where Set indicates the category of sets.
- $\mathcal{F}$ is the full subcategory of all finitely presentable objects in $\mathcal{V}$. This category $\mathcal{F}$ is closed in $\mathcal{V}$ under finite colimits and every object of $\mathcal{F}$ is

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the coequalizer of two morphims in $\mathcal{P}$. The category $\mathcal{V}$ is also equivalent to the category of finite limit preserving functors $\mathcal{F}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ and every object in $\mathcal{V}$ is the colimit of a filtered diagram in $\mathcal{F}$.

- $\mathsf{Flat}(\mathcal{V})$ is the full subcategory of flat objects in $\mathcal{V}$, that is, those objects which can be presented as the colimit of a filtered diagram in $\mathcal{P}$, not just in $\mathcal{F}$ (see [14]).

Let us also recall that an object X of $\mathcal{V}$ is finitely generated when the representable functor $\mathcal{V}(X, -)$ preserves filtered unions of subobjects; this is equivalent to X being a regular quotient of some object of $\mathcal{F}$. For the sake of clarity, we reserve

- letters A, B, C, ... for the objects of $\mathcal{P}$,
- letters K, L, M, ... for the objects of $\mathcal{F}$,
- letters P, Q, R, ... for objects of $\mathsf{Flat}(\mathcal{V})$,
- letters ..., X, Y, Z for arbitrary objects of $\mathcal{V}$.

Finally, a monomorphism $s\colon X \longrightarrow Y$ of $\mathcal{V}$ is pure when, given a commutative square $h \circ s = f \circ g$ with $M, N \in \mathcal{F}$,

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[d, "h"] \arrow[dl, dotted, "\varphi"'] \\ X \arrow[r, "s"'] & Y \end{tikzcd}$$

there exists a morphism φ such that $\varphi \circ f = g$. This is equivalent to s being a filtered colimit of split monomorphisms (monomorphisms with a retraction) (see [2] and [15]).

In the case of the variety $\mathcal{V}$ of left modules over a ring R with unit, all categorical notions recalled above are equivalent to the usual algebraic ones (see [2], [15] and [10]). In particular, a left R-module Z is flat iff for every monomorphism $f\colon X \rightarrowtail Y$ of right R-modules, the mapping

$$X \otimes_R Z \xrightarrow{f \otimes Z} Y \otimes_R Z$$

is still injective. A monomorphism $f\colon X \longrightarrow Y$ of left R-modules is pure iff for every right R-module Z, the mapping

$$Z \otimes_R X \xrightarrow{Z \otimes f} Z \otimes_R Y$$

is still injective.

For a ring R with unit, the following conditions are well-known to be equivalent (see [16]):

1. $\forall r \in R \;\; \exists s \in R \;\; r = rsr$;

2. every left R-module is flat;

3. every monomorphism of left R-modules is pure;

4. every finitely generated left ideal of R is a direct summand.

Of course the result could be stated equivalently for right R-modules, since the first condition is left-right symmetric. Such rings are called *von Neumann* regular rings. Let us recall that a category of modules is abelian: in particular it is additive, regular and even exact; every epimorphism is regular and every monomorphism is regular; retracts coincide with direct summands.

The main purpose of this paper is to generalize these considerations to the case of an arbitrary variety, by investigating the force of the von Neumann axiom in $\mathcal{P}$ or in $\mathcal{F}$. Of course the von Neumann axiom makes obvious sense in every category:

$$\forall r\colon A \to B \;\; \exists s\colon B \to A \quad r = r \circ s \circ r.$$

We shall prove in particular that the von Neumann axiom in $\mathcal{F}$ is equivalent to every object of $\mathcal{V}$ being flat and every monomorphism in $\mathcal{V}$ being pure, generalizing so the situation for modules. On the other hand, the von Neumann axiom in $\mathcal{P}$ is equivalent to every subobject of a flat object being pure and is further equivalent to every finitely generated subobject of an object in $\mathcal{P}$ being a retract. The stronger property that every finitely generated subobject of an object in $\mathcal{F}$ (not just in $\mathcal{P}$) is a retract is in fact equivalent to every subobject in $\mathcal{V}$ being pure.

In the abelian case, the situation recaptures even more flavor of the classical case of modules. The flatness of every object is now equivalent to the purity of every monomorphism and is further equivalent to the von Neumann axiom in $\mathcal{P}$; this forces $\mathcal{P} = \mathcal{F}$. As a consequence, the classical von Neumann axiom on a ring R implies the same axiom in the whole category of finitely presentable left (or right) R-modules. This was proved by B. Stenström (see [17], even for Grothendieck categories). Let us recall that abelian varieties are precisely the categories of additive functors $\mathsf{Add}[\mathcal{C}, \mathsf{Ab}]$ where $\mathcal{C}$ is a small additive category.

Finally, we investigate also conditions under which the category $\mathsf{Flat}(\mathcal{V})$ of flat objects is reflective in $\mathcal{V}$: this is equivalent to the canonical theory $\mathcal{P}^{\mathsf{op}}$ having finite limits, not just finite products. We consider also the case where $\mathsf{Flat}(\mathcal{V})$ is merely weakly reflective in $\mathcal{V}$. We conclude with some examples and counterexamples.

1 Closenedness properties of $\mathcal{F}$

As recalled in the introduction, the category $\mathcal{P}$ is closed in $\mathcal{V}$ under finite coproducts, thus the inclusion $\mathcal{P} \subseteq \mathcal{V}$ preserves and reflects finite coproducts. Analogously $\mathcal{F}$ is closed in $\mathcal{V}$ under finite colimits, thus the inclusion $\mathcal{F} \subseteq \mathcal{V}$ preserves and reflects finite colimits. This implies at once

1 **Proposition 1** *The inclusion $\mathcal{F} \subseteq \mathcal{V}$ preserves and reflects epimorphisms.*

Proof In an arbitrary category, a morphism $f: M \longrightarrow N$ is an epimorphism iff its cokernel pair is $(\mathsf{id}_N, \mathsf{id}_N)$. From which the result, since $\mathcal{F} \subseteq \mathcal{V}$ preserves and reflects cokernel pairs. $\square$

2 **Proposition 2** *The inclusion $\mathcal{F} \subseteq \mathcal{V}$ preserves and reflects monomorphisms.*

Proof In an arbitrary category with a family $\mathcal{G}$ of generators, a morphism $f: M \longrightarrow N$ is a monomorphism when

$$\forall K \in \mathcal{G} \;\; \forall u, v: K \rightrightarrows M \quad f \circ u = f \circ v \Rightarrow u = v.$$

From which the conclusion, since the objects of $\mathcal{F}$ constitute a set of generators in $\mathcal{V}$. $\square$

Our next result is much more subtle. The inclusion $\mathcal{F} \subseteq \mathcal{V}$ preserves coequalizers, thus the fact of being a regular epimorphism. But this inclusion has no "on the nose" reason to reflect regular epimorphisms: if a morphism $f: M \longrightarrow N$ of $\mathcal{F}$ is the coequalizer of two morphims $g, h: X \rightrightarrows M$ in $\mathcal{V}$, why should it also be the coequalizer of two morphisms in $\mathcal{F}$? Indeed $\mathcal{F}$ is generally not finitely complete, thus the canonical choice of the kernel pair of f does not help. Nevertheless:

3 **Theorem 3** *The inclusion $\mathcal{F} \subseteq \mathcal{V}$ preserves and reflects regular epimorphisms.*

Proof The inclusion preserves coequalizers, thus regular epimorphisms. Conversely, let $f: M \longrightarrow N$ be a morphism of $\mathcal{F}$ which is a regular epimorphism in $\mathcal{V}$. Thus f is the coequalizer of two morphisms $g, h: X \to M$ in $\mathcal{V}$. We express X as a filtered colimit $(l_i: L_i \longrightarrow X)_{i \in I}$ of objects $L_i \in \mathcal{F}$ and form the coequalizers $f_i = \mathsf{Coker}\,(g \circ l_i, h \circ l_i)$ in $\mathcal{F}$.

$$\begin{tikzcd}
L_i \arrow[r, "l_i"] & X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & M \arrow[r, two heads, "f_i"] \arrow[dr, "f"'] & N_i \arrow[d, "n_i"] \\
 & & & N
\end{tikzcd}$$

We obtain a unique factorization n_i, for each index $i \in I$. Obviously, every morphism $\lambda_{i,j}: L_i \longrightarrow L_j$ in the filtered diagram with colimit X yields a corresponding factorization $\nu_{i,j}: N_i \longrightarrow N_j$ between the coequalizers. This turns the coequalizers N_i in a filtered diagram in $\mathcal{F}$ and an obvious diagram chasing shows that $\mathsf{colim}_{i \in I} N_i = N$. Since $N \in \mathcal{F}$, the identity on N factors through one term of this colimit: thus there is $i_0 \in I$ and $t: N \longrightarrow N_{i_0}$ such that $n_{i_0} \circ t = \mathsf{id}_N$. Since

$$n_{i_0} \circ t \circ f = f = n_{i_0} \circ f_{i_0},$$

there is $j \geq i_0$ such that

$$\nu_{i_0,j} \circ t \circ f = \nu_{i_0,j} \circ f_{i_0} = f_j.$$

We have then

$$n_j \circ (\nu_{i_0,j} \circ t) = n_{i_0} \circ t = \mathsf{id}_N$$

and

$$(\nu_{i_0,j} \circ t) \circ n_j \circ f_j = \nu_{i_0,j} \circ t \circ f = f_j.$$

Hence

$$(n_{i_0,j} \circ t) \circ n_j = \mathsf{id}_{N_j}$$

because f_j is an epimorphism. Thus $n_{i_0,j} \circ t = n_j^{-1}$ and f is isomorphic, via n_j, to the coequalizer f_j in $\mathcal{F}$. $\square$

2 About global purity

This section investigates the conditions under which every monomorphism of $\mathcal{V}$ is pure.

$\boxed{5}$ **Theorem 4** *The following conditions are equivalent in a variety $\mathcal{V}$:*

1. *every monomorphism of $\mathcal{V}$ is pure;*

2. *in $\mathcal{F}$, every arrow factors as a regular epimorphism followed by a split monomorphism;*

3. *in $\mathcal{V}$, every finitely generated subobject of a finitely presentable object is a retract.*

Proof In view of proposition 3, there is no ambiguity on the meaning of condition 2.

(1) $\Rightarrow$ (2). Consider a morphism $f\colon M \longrightarrow N$ in $\mathcal{F}$, factor it through its image $f = i \circ p$ in the regular category $\mathcal{V}$ and consider the following situation:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, two heads, "p"'] & N \arrow[d, equal] \arrow[dl, dotted, "\varepsilon"'] \\ I \arrow[r, tail, "i"'] & N \end{tikzcd}$$

By purity of i, we get ε such that $\varepsilon f = p$. This implies $\varepsilon i p = \varepsilon f = p$, thus $\varepsilon i = \mathsf{id}_I$ because p is an epimorphism. Thus I is a retract of $N \in \mathcal{F}$ and by Cauchy completeness of $\mathcal{F}$ (it is finitely cocomplete), we get $I \in \mathcal{F}$.

(2) $\Rightarrow$ (3). Consider a monomorphism $m\colon X \rightarrowtail M$ in $\mathcal{V}$, with $M \in \mathcal{F}$ and X finitely generated. By assumption on X, there is a regular epimorphism

$p\colon A \twoheadrightarrow X$ in $\mathcal{V}$, with $A \in \mathcal{P}$. The morphism $f = mp\colon A \longrightarrow M$ of $\mathcal{F}$ admits thus the image factorization $m \circ p$; by assumption 2 and the uniqueness of the image factorization, m is a split mono.

(3) $\Rightarrow$ (1). Consider a monomorphism s in $\mathcal{V}$ and a commutative square $hf = sg$, with f in $\mathcal{F}$ (see diagram 1). Factor f through its image $f = i \circ p$ in $\mathcal{V}$.

$$\begin{tikzcd}
M \arrow[r, two heads, "p"] \arrow[rr, bend left, "f"] \arrow[d, "g"'] & Z \arrow[r, tail, "i"] \arrow[d, "t"'] & N \arrow[l, bend left, "r"] \arrow[d, "h"] \\
X \arrow[r, equal] \arrow[rr, bend right, "s"'] & X \arrow[r, tail, "s"] & Y
\end{tikzcd}$$

Diagram 1

Then Z is finitely generated as a regular quotient of $M \in \mathcal{F}$. By assumption 3, i admits a retraction r. By functoriality of the image construction, there is t such that $tp = g$ and $hi = st$. Finally

$$strf = hirf = hirip = hip = hf = sg$$

from which $trf = g$ since s is a monomorphism. Thus s is pure. $\square$

$\boxed{8}$ **Corollary 5** *In a variety where all monomorphisms are pure,*

1. *all epimorphisms between finitely presentable objects are regular;*
2. *all monomorphisms between finitely presentable objects are split.*

Proof Let $f\colon M \longrightarrow N$ be a morphism in $\mathcal{F}$, which by theorem 4 factors as $f = ip$ with p a regular epimorphism and i a split monomorphism. When f is an epimorphism, i is an epimorphism as well and thus an isomorphism; so f is isomorphic to the regular epimorphism p. When f is a monomorphism, p is a monomorphism as well and thus an isomorphism; thus f is isomorphic to the split mono i. $\square$

Since a pure monomorphism is necessarily regular, it makes sense to consider the more general case where only regular monomorphisms are requested to be pure. For this we need the following lemma, which is part of the "folklore":

$\boxed{22}$ **Lemma 6** *Let $\mathcal{C}$ be a well-powered and complete category. When the composite of two regular monomorphisms is again a regular monomorphism, every morphism of $\mathcal{C}$ factors as an epimorphism followed by a regular monomorphism. Such a factorization is unique up to isomorphism and is functorial.*

Proof Given a morphism $f\colon X \longrightarrow Y$, one factors it as $f = i \circ p$ through the intersection $i\colon I \rightarrowtail Y$ of all regular subobjects of Y through which f factors. This is still a regular subobject, as intersection of regular subobjects. The corresponding factorization $p\colon X \longrightarrow I$ of f, yielding $f = ip$, is necessarily an epimorphism. Indeed $up = vp$ implies that p factors through the equalizer $k\colon K \rightarrowtail I$ of (u, v). But then f factors through the regular monomorphism $i \circ k$, thus $I \subseteq K$ and k is an isomorphism. This implies $u = v$. The rest is routine. $\square$

23 **Theorem 7** *The following conditions are equivalent in a variety $\mathcal{V}$:*

1. *in $\mathcal{V}$, every regular monomorphism is pure;*
2. *in $\mathcal{F}$, every arrow factors as an epimorphism followed by a split monomorphism.*

Proof In view of proposition 1, there is no ambiguity on the meaning of condition 2.

(1) $\Rightarrow$ (2). The composite of two pure monomorphisms is pure. So the assumption forces the validity in $\mathcal{V}$ of the conditions of lemma 6. Now simply repeat the proof of the corresponding assertion in theorem 4, with this time p an epimorphism and i a regular monomorphism.

(2) $\Rightarrow$ (1). Again it suffices to adapt the argument concerning diagram 1 in theorem 4: s is now a regular monomorphism and $i \circ p$ is the factorization of f given at once by condition 2, with r a retraction of i. If $s = \mathsf{Ker}\,(u, v)$, then $uhip = usg = vsg = vhip$, thus $uhi = vhi$ because p is an epimorphism. This implies the existence of a factorization t through $\mathsf{Ker}\,(u, v)$, yielding $st = hi$. As a consequence, $stp = hip = hf = sg$, thus $tp = g$ because s is a monomorphism. As in theorem 4, tr is the expected morphism. $\square$

To investigate a first implication between purity and flatness properties, let us recall another notion (see [3]):

50 **Definition 8** *In a variety $\mathcal{V}$, a regular epimorphism $p\colon X \twoheadrightarrow Y$ is a pure quotient when every arrow $f\colon M \longrightarrow Y$ with M finitely presentable admits a factorization through p:*

$$\begin{tikzcd} & M \arrow[dl, dotted, "g"'] \arrow[d, "f"] \\ X \arrow[r, two heads, "p"'] & Y \end{tikzcd}$$

In a variety $\mathcal{V}$, the pure quotient maps are exactly the filtered colimits of split epimorphisms. Notice that no confusion can occur with the notion of pure monomorphism, since a morphism which is both a monomorphism and a regular epimorphism is necessarily an isomorphism; therefore, it is trivially both a pure monomorphism and a pure epimorphism.

18bis **Theorem 9** *In a variety $\mathcal{V}$, the following conditions are equivalent:*

1. *every algebra is flat;*
2. *every finitely presentable object is projective ($\mathcal{P} = \mathcal{F}$);*
3. *every regular epimorphism in $\mathcal{F}$ splits;*
4. *every regular epimorphism in $\mathcal{V}$ is pure.*

Proof (1) ⇒ (2). Every $M \in \mathcal{F}$ is flat by assumption, thus a filtered colimit $M = \mathsf{colim}\, B_i$ with $B_i \in \mathcal{P}$. By finite presentability of M, the identity on M factors through some B_i, thus $M \in \mathcal{P}$ because M is a retract of this $B_i \in \mathcal{P}$.

(2) ⇒ (3) is trivial and condition (4) is just rephrasing condition (2).

(3) ⇒ (2). Every $M \in \mathcal{F}$ can be written as a regular quotient $p\colon A \twoheadrightarrow M$, with $A \in \mathcal{P}$. Then p splits by assumption and $M \in \mathcal{P}$ because it is a retract of $A \in \mathcal{P}$.

(2) ⇒ (1) is obvious, since then every algebra is the colimit of a filtered diagram in $\mathcal{F} = \mathcal{P}$. □

Remark. The fact that every regular epimorphism in $\mathcal{V}$ is a filtereded colimit of regular epimorphisms in $\mathcal{V}$ between objects in $\mathcal{F}$ is in [8] (lemma 12). But the claim is stronger there and, in fact, the authors have forgotten to prove our Theorem 3.

When the variety $\mathcal{V}$ is pointed (i.e. admits a zero object), there is an even stronger notion of monomorphism: the normal monomorphisms. A monomorphism is normal when it is the kernel of some arrow, that is, the equalizer of an arrow and the zero arrow. The semi-abelian categories constitute a situation where normal subobjects have good properties (see [12] and [7]).

26 **Lemma 10** *Let $\mathcal{V}$ be a semi-abelian variety in which every normal monomorphism is pure. Then every regular epimorphism is a pure quotient.*

Proof In a semi-abelian category, every regular epimorphism is the cokernel of its kernel (see [7]). This kernel is a pure monomorphism by assumption. But in a semi-abelian variety, every cokernel of a pure normal subobject is a pure quotient (see [3]). □

27 **Theorem 11** *Let $\mathcal{V}$ be a semi-abelian variety. If every monomorphism is pure, every algebra is flat.*

Proof By lemma 10 and theorem 9. □

3 The von Neumann axiom

First, let us introduce and study the von Neumann axiom in a very general setting.

13Law **Definition 12** *A morphism $f\colon A \longrightarrow B$ in a category $\mathcal{C}$ splits when there exists a morphism $g\colon B \longrightarrow A$ such that $f \circ g \circ f = f$.*

Of course, when f is a monomorphism or an epimorphism, this definition reduces at once to the usual definition of a split monomorphism or a split epimorphism.

14Law **Lemma 13** *Let $f\colon A \longrightarrow B$ be a morphism in a category $\mathcal{C}$. The following conditions are equivalent:*

1. *the morphism f splits;*
2. *there exists a morphism $h\colon B \longrightarrow A$ such that $fhf = f$ and $hfh = h$.*

Proof Given g as in definition 12, simply put $h = gfg$. □

13 **Definition 14** *A category $\mathcal{C}$ satisfies the von Neumann axiom when every morphism of $\mathcal{C}$ splits.*

14 **Proposition 15** *In a Cauchy complete category $\mathcal{C}$, the following conditions are equivalent:*

1. *$\mathcal{C}$ satisfies the von Neumann axiom;*
2. *every arrow of $\mathcal{C}$ factors as a split epimorphism followed by a split monomorphism.*

Proof (1) $\Rightarrow$ (2). Consider $f\colon A \longrightarrow B$. Choose $g\colon B \longrightarrow A$ such that $fgf = f$. Then $fgfg = fg$, thus $fg\colon B \longrightarrow B$ is an idempotent. By Cauchy completeness of $\mathcal{C}$, fg splits, yielding $fg = iq$ where $qi = \mathsf{id}_I$. This gives the diagram

$$\begin{tikzcd} A \arrow[rr, shift right, "f"'] \arrow[dr, shift right, "qf"'] & & B \arrow[ll, shift right, "g"'] \arrow[dl, shift right, "q"'] \\ & I \arrow[ul, shift right, "gi"'] \arrow[ur, shift right, "i"'] & \end{tikzcd}$$

where $(qf)(gi) = q(fg)i = qiqi = \mathsf{id}_I$, while $iqf = iqiq = iq = f$.

(2) $\Rightarrow$ (1) Consider $f\colon A \longrightarrow B$ and write it as the composite of a split epimorphism p and a split monomorphism i: $f = ip$, $ps = \mathsf{id}_I = qi$.

$$\begin{tikzcd} A \arrow[rr, shift right, "f"'] \arrow[dr, shift right, "p"'] & & B \arrow[ll, shift right, "sq"'] \arrow[dl, shift right, "q"'] \\ & I \arrow[ul, shift right, "s"'] \arrow[ur, shift right, "i"'] & \end{tikzcd}$$

It follows at once that $f(sq)f = ipsqip = ip = f$. □

The relevance of the von Neumann axiom in the study of pure subobjects is emphasized by the following two theorems. To facilitate the language, we introduce an intermediate notion.

$\boxed{28}$ **Definition 16** *In a variety $\mathcal{V}$, a monomorphism $s\colon X \rightarrowtail Y$ is $\mathcal{P}$-pure when given a commutative square $hf = sg$ with $A, B \in \mathcal{P}$,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, dotted, "\varepsilon"'] \\ X \arrow[r, tail, "s"'] & Y \end{tikzcd}$$

there exists a morphism ε such that $\varepsilon f = g$.

$\boxed{21}$ **Theorem 17** *In a variety $\mathcal{V}$, the following conditions are equivalent:*

1. *the von Neumann axiom in $\mathcal{P}$;*
2. *every monomorphism of $\mathcal{V}$ is $\mathcal{P}$-pure;*
3. *every subobject of a flat object is pure;*
4. *every finitely generated subobject of a finitely presentable projective object is a retract.*

Proof (1) $\Rightarrow$ (2). Given a commutative square as in definition 16, choose $\varphi\colon B \longrightarrow A$ such that $f\varphi f = f$. Then $sg\varphi f = hf\varphi f = hf = sg$, thus $g\varphi f = g$ since s is a monomorphism. So $\varepsilon = g\varphi$ is the expected morphism.

(2) $\Rightarrow$ (3). Consider a monomorphism $s\colon X \rightarrowtail P$ in $\mathcal{V}$, with P flat. Consider a commutative square $hf = sg$, with $M, N \in \mathcal{F}$. Choose a regular epimorphism $p\colon A \twoheadrightarrow M$, with $A \in \mathcal{P}$. Write P as a filtered colimit $P = \mathsf{colim}\, B_i$, with $B_i \in \mathcal{P}$. Since N is finitely presentable, h factors through one of the B_i, yielding $h = \sigma_i t$.

$$\begin{tikzcd} A \arrow[r, two heads, "p"] & M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[r, "t"] \arrow[d, "h"] & B_i \arrow[dl, "\sigma_i"] \arrow[dll, dotted, "\varphi"'] \\ & X \arrow[r, tail, "s"'] & P & \end{tikzcd}$$

By assumption s is $\mathcal{P}$-pure, from which a morphism φ such that $\varphi tfp = gp$. Since p is an epimorphism, this implies $\varphi tf = g$, thus $\varepsilon = \varphi t$ is the expected morphism.

(3) $\Rightarrow$ (4). Consider a finitely generated subobject $s\colon X \rightarrowtail B$ with $B \in \mathcal{P}$; in particular, B is flat. By assumption on X, there exists a regular epimorphism $p\colon A \twoheadrightarrow X$ with $A \in \mathcal{P}$. Considering the diagram

$$\begin{tikzcd} A \arrow[r, "sp"] \arrow[d, two heads, "p"'] & B \arrow[d, equal] \arrow[dl, dotted, "\varepsilon"'] \\ X \arrow[r, tail, "s"'] & B \end{tikzcd}$$

the purity of s implies the existence of ε such that $\varepsilon sp = p$. Since p is an epimorphism, $\varepsilon s = \mathsf{id}_X$.

(4) $\Rightarrow$ (1). Consider a morphism $f\colon A \longrightarrow B$ in $\mathcal{P}$ and its image factorization $s \circ p$ in $\mathcal{V}$, with image object X.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & B \\ & X \arrow[ur, tail, "s"'] & \end{tikzcd}$$

Since p is a regular epimorphism, X is finitely generated; thus s is a retract by assumption. But then $X \in \mathcal{P}$, as a retract of $B \in \mathcal{P}$. In particular X is projective and p splits. One concludes by proposition 15. $\square$

29 **Theorem 18** *In a variety $\mathcal{V}$, the following conditions are equivalent:*

1. *the von Neumann axiom in $\mathcal{F}$;*
2. *every object of $\mathcal{V}$ is flat and every monomorphism of $\mathcal{V}$ is pure.*

In those conditions, every finitely presentable algebra is projective ($\mathcal{F} = \mathcal{P}$).

Proof (1) $\Rightarrow$ (2). By theorem 17, it suffices to prove that every object is flat. For this it suffices to prove that $\mathcal{P} = \mathcal{F}$, that is, every finitely presentable object is projective (see theorem 9). A finitely presentable object $M \in \mathcal{F}$ is a regular quotient $p\colon A \twoheadrightarrow M$ of an object $A \in \mathcal{P}$. By the von Neumann axiom in $\mathcal{F}$, choose $s\colon M \longrightarrow A$ such that $psp = p$. Since p is an epimorphism, $ps = \mathsf{id}_M$. Thus M is projective, as a retract of the projective object A.

(2) $\Rightarrow$ (1). Again by theorem 9, every object of $\mathcal{V}$ is flat precisely when $\mathcal{P} = \mathcal{F}$. Thus condition 2, together with theorem 17, implies the von Neumann axiom in $\mathcal{P} = \mathcal{F}$. $\square$

NL **Lemma 19** *Let $\mathcal{C}$ be a category with finite coproducts and a zero object. The following conditions are equivalent:*

1. *$\mathcal{C}$ satisfies the von Neumann axiom;*
2. *for each object $C \in \mathcal{C}$, the monoids $\mathsf{Hom}(C, C)$ satisfy the von Neumann axiom.*

Proof Necessity is evident. Assume now that all monoids $\mathsf{Hom}(C, C)$, $C \in \mathcal{C}$, satisfy the von Neumann axiom and consider a morphism $f\colon C \longrightarrow D$ in $\mathcal{C}$. Take

$$C \amalg D \xrightarrow{(\mathsf{id}_C, 0)} C \xrightarrow{\quad f \quad} D \xrightarrow{\quad i_D \quad} C \amalg D.$$

By the von Neumann axiom in $\mathsf{Hom}(C \amalg D, C \amalg D)$ there is $r\colon C \amalg D \longrightarrow C \amalg D$ such that

$$\big(i_D \circ f \circ (\mathsf{id}_C, 0)\big) \circ r \circ \big(i_D \circ f \circ (\mathsf{id}_C, 0)\big) = i_D \circ f \circ (\mathsf{id}_C, 0).$$

Composing on the left with $(0, \mathsf{id}_D)$ and on the right with i_C yields

$$f \circ \Big((\mathsf{id}_C, 0) \circ r \circ i_D\Big) \circ f = f.$$

Therefore the category $\mathcal{C}$ satisfies the von Neumann axiom. $\square$

4 The abelian case

Finally, we specialize our results on purity in the special context of abelian varieties, to arrive finally at a situation analogous to that of modules on a von Neumann regular ring.

Let us recall that in an abelian variety $\mathcal{V}$, every split subobject is a direct summand. In particular, the cokernel of a split monomorphism is a split epimorphism (the other leg of the direct sum). By duality, the kernel of a split epimorphism is a split monomorphism. Moreover, every monomorphism is the kernel of its cokernel, and dually.

PSQ

Proposition 20 *Consider a short exact sequence*

$$0 \longrightarrow A \overset{k}{\rightarrowtail} B \overset{q}{\twoheadrightarrow} C \longrightarrow 0$$

in an abelian variety $\mathcal{V}$. The following conditions are equivalent:

1. *k is a pure monomorphism;*
2. *q is a pure epimorphism.*

Proof Pure subobjects are the filtered colimits of split monomorphisms (see proposition 2.30 in [2]). Pure quotients are the filtered colimits of split epimorphisms (see proposition 4 in[3]). But in an abelian variety, cokernels of split monomorphisms are split epimorphisms, and dually. Moreover, filtered colimits commute with both kernels and cokernels. This implies at once the result. $\square$

In the abelian case, which contains the case of modules on a ring (and reduces to it in the one-sorted case), we get thus the "classical" situation well-known for von Neumann rings:

51 **Theorem 21 (Stenström)** *Let $\mathcal{V}$ be an abelian variety. The following conditions are equivalent:*

1. *the von Neumann axiom in $\mathcal{P}$;*
2. *every monomorphism is pure;*
3. *every epimorphism is pure;*
4. *every algebra is flat;*
5. *every finitely presentable algebra is projective ($\mathcal{F} = \mathcal{P}$);*
6. *every finitely generated subobject of a finitely presentable object is a direct summand;*
7. *for every $P \in \mathcal{P}$, $\mathsf{Hom}(P, P)$ is a von Neumann ring.*

In particular, $\mathcal{F}$ satisfies the von Neumann axiom.

Proof (1) ⇔ (6) is attested by theorem 17. (2) ⇔ (3) follows at once from proposition 20, since every monomorphism is the kernel of its cokernel, and dually. (3) ⇔ (4) ⇔ (5) hold by theorem 9. (2) ⇔ (6) holds by theorem 4. (1) ⇔ (7) by lemma 19. □

It is well-known that a ring R is a von Neumann ring precisely when the corresponding variety $\mathcal{V}$ of R-modules satisfies one of the conditions 2 to 6 in theorem 21 (see [17]). In that case the corresponding category $\mathcal{P} = \mathcal{F}$ satisfies thus the von Neumann axiom, but certainly not the whole variety $\mathcal{V}$ of R-modules. Going back to the comments following definition 12, observe that this would in particular imply that all monomorphisms are split and all epimorphisms are split: this is not the case in general; compare with condition 6 in theorem 21.

5 On the flat reflection

Our next theorem is the flat reflection theorem. Let us make a strong point that $\mathcal{P}$ is closed in $\mathcal{F}$ and $\mathcal{V}$ under finite coproducts, but not necessarily under coequalizers, even when these turn out to exist.

24 **Theorem 22** *In a variety $\mathcal{V}$, the following conditions are equivalent:*

1. *$\mathcal{P}$ is finitely cocomplete;*
2. *$\mathcal{P}$ is reflective in $\mathcal{F}$;*
3. *$\mathsf{Flat}(\mathcal{V})$ is reflective in $\mathcal{V}$;*
4. *$\mathsf{Flat}(\mathcal{V})$ is closed under limits in $\mathcal{V}$.*

Proof (1) ⇒ (2). Consider $M \in \mathcal{F}$ and write it as a coequalizer $p = \mathsf{Coker}\,(u, v)$ in $\mathcal{V}$ of two morphisms $u, v \in \mathcal{P}$.

$$\begin{tikzcd}
A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, two heads, "p"] \arrow[dr, "q"'] & M \arrow[d, "r"] \arrow[dr, "f"] & \\
& & C \arrow[r, dotted, "g"'] & D
\end{tikzcd}$$

If $q = \mathsf{Coker}\,(u, v)$ in $\mathcal{P}$, $qu = qv$ implies the existence of $r \in \mathcal{V}$ such that $rp = q$. Let us prove that $r\colon M \longrightarrow C$ yields the reflection of M in $\mathcal{P}$.

Given $f\colon M \longrightarrow D$ with $D \in \mathcal{P}$, $fpu = fpv$ implies the existence of a unique $g \in \mathcal{P}$ such that $gq = fp$, because $q = \mathsf{Coker}\,(u, v)$ in $\mathcal{P}$. This implies $grp = gq = fp$, thus $gr = f$ since p is an epimorphism in $\mathcal{V}$. If $g'\colon C \longrightarrow D$ is another factorization yielding $g'r = f$, then $g'q = g'rp = fp = gq$, thus $g = g'$ because q is an epimorphism in $\mathcal{P}$.

(2) ⇒ (3). $\mathsf{Flat}(\mathcal{V})$ is the so-called Ind-completion of $\mathcal{P}$, that is, the closure of $\mathcal{P}$ in $[\mathcal{P}^{\mathsf{op}}, \mathsf{Set}]$ under filtered colimits (see [11] or [2]). Analogously, $\mathcal{V}$ is the Ind-completion of $\mathcal{F}$: the closure of $\mathcal{F}$ in $[\mathcal{F}^{\mathsf{op}}, \mathsf{Set}]$ under filtered colimits. This Ind-completion process is trivially 2-functorial, that is extends to functors and natural transformations. Since an adjunction can be presented equationally via its unit, its counit and the two triangular identities, the Ind-completion process transforms an adjunction in an adjunction. Thus writing

$$r \dashv i\colon \mathcal{P} \leftrightarrows \mathcal{F}$$

for the adjunction in condition 2, we get an adjunction

$$\mathsf{Ind}(r) \dashv \mathsf{Ind}(i)\colon \mathsf{Flat}(\mathcal{V}) \leftrightarrows \mathcal{V}.$$

Of course, $\mathsf{Ind}(i)$ is simply the canonical inclusion $\mathsf{Flat}(\mathcal{V}) \hookrightarrow \mathcal{V}$.

(3) ⇒ (4) holds for every reflective subcategory (see [6]) .

(4) ⇒ (3) is the fact that every accessible functor between locally presentable categories admits a left adjoint as soon as it preserves limits (see [2], theorem 2.48).

(3) ⇒ (2). $\mathsf{Flat}(\mathcal{V})$ is closed in $\mathcal{V}$ under filtered colimits, which is known to be equivalent to the reflection $\mathcal{V} \longrightarrow \mathsf{Flat}(\mathcal{V})$ preserving finite presentability (see [11] or [2]). Thus the adjunction in condition 3 restricts to an adjunction between $\mathcal{P}$ and $\mathcal{F}$.

(2) ⇒ (1). A reflective subcategory of a finitely cocomplete category is always finitely cocomplete. □

The following refinement of theorem 22 is also of interest:

25 **Theorem 23** *In variety $\mathcal{V}$, the following conditions are equivalent:*

1. *$\mathcal{P}$ admits coequalizers and these are split epimorphisms;*

2. $\mathcal{P}$ *is split-epireflective in* $\mathcal{F}$
   (at each object, the unit of the adjunction is a split epimorphism);

3. $\mathsf{Flat}(\mathcal{V})$ *is regular-epireflective in* $\mathcal{V}$
   (at each object, the unit of the adjunction is a regular epimorphism);

4. $\mathsf{Flat}(\mathcal{V})$ *is pure-epireflective in* $\mathcal{V}$*;*
   (at each object, the unit of the adjunction is a pure quotient).

Proof (1) $\Rightarrow$ (2). By theorem 22 and with the notation of its proof of (1) $\Rightarrow$ (2), it remains to show that r is a split epimorphism. By assumption, q admits a section s. Then $rpsq = qsq = q$, thus $rps = \mathsf{id}_C$ since q is an epimorphism in $\mathcal{P}$.

(2) $\Rightarrow$ (4). As observed in the proof of theorem 22, the unit of the adjunction $\mathsf{Flat}(\mathcal{V}) \leftrightarrows \mathcal{V}$ is $\mathsf{Ind}(\eta)$, where η is the unit of the adjunction $\mathcal{P} \leftrightarrows \mathcal{F}$. By assumption, $\mathsf{Ind}(\eta)$ is thus a filtered colimit of split epimorphisms, that is a pure quotient.

(4) $\Rightarrow$ (3) is trivial, because every pure quotient is regular.

(3) $\Rightarrow$ (2). We have observed in the proof of theorem 22 that the adjunction in condition 3 restricts to an adjunction between $\mathcal{P}$ and $\mathcal{F}$. Given $M \in \mathcal{F}$, the unit $\eta_M \colon M \longrightarrow C$ is a regular epimorphism by assumption, thus a split epimorphism since C is projective.

(2) $\Rightarrow$ (1). $\mathcal{P}$ is finitely cocomplete as reflective subcategory of the finitely cocomplete $\mathcal{F}$. Now consider $q = \mathsf{Coker}\,(u, v)$ in $\mathcal{P}$ and compute $p = \mathsf{Coker}\,(u, v)$ in $\mathcal{V}$.

$$\begin{tikzcd} A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, two heads, "p"] \arrow[dr, shift right, "q"'] & M \arrow[d, shift left, "r"] \\ & & C \arrow[ul, shift right, "t"'] \arrow[u, shift left, "s"] \end{tikzcd}$$

This yields a factorization $rp = q$ in $\mathcal{V}$ and we have seen in the proof of theorem 22 that $r \colon M \longrightarrow C$ is the flat reflection of M. By assumption, r admits a section s. But since C is projective, s factors through p as $s = pt$. This yields $qt = rpt = rs = \mathsf{id}_C$, proving that q is a split epimorphism. $\square$

$\boxed{ex}$ **Examples 24**

1. Let $\mathcal{V}$ be the variety of all left R-modules. Then $\mathsf{Flat}(\mathcal{V})$ is reflective iff R is right coherent and of weak global dimension at most two (see [9], 8.4.29). Let us mention that $\mathsf{Flat}(\mathcal{V})$ is epireflective iff R is right coherent and of weak global dimension at most one (see again [9], Ex. 5, page 139).
2. Let $\mathcal{C}$ be a small category and consider the variety $\mathcal{V} = \mathsf{Set}^{\mathcal{C}^{op}}$. Then $\mathcal{P} = \mathsf{Fam}(\mathcal{C})$ is the free completion of $\mathcal{C}$ under finite coproducts and $\mathsf{Flat}(\mathcal{V}) = \mathsf{Ind}\big(\mathsf{Fam}(\mathcal{C})\big)$. Hence $\mathsf{Flat}(\mathcal{V})$ is reflective in $\mathcal{V}$ iff $\mathsf{Fam}(\mathcal{C})$ is finitely cocomplete. This is equivalent to $\mathcal{C}$ having finite connected colimits. $\square$

6 On weak reflections

We consider now a weaker situation than in section 5: the case where the category of flat algebras is only weakly reflective in the category of all algebras (see [2] for an introduction to weak reflections and weak colimits). Since our arguments hold in a much more general context and do not seem to have been explicitely observed before, we present them in full generality. Given a small category $\mathcal{F}$, we write as usual $\mathsf{Ind}(\mathcal{F})$ for the closure of $\mathcal{F}$ in $[\mathcal{F}^{\mathsf{op}}, \mathsf{Set}]$ under filtered colimits, that is, the $\aleph_0$-accessible category of flat functors on $\mathcal{F}$. We are interested in the case where $\mathsf{Ind}(\mathcal{F})$ is weakly locally presentable, that is, admits products (see theorem 4.11 in [2]).

$\boxed{WRL}$ **Proposition 25** *Let $\mathcal{K}$ be an $\aleph_0$-accessible category. The following conditions are equivalent:*

1. *$\mathcal{K}$ admits products;*
2. *$\mathcal{K}$ is equivalent to $\mathsf{Ind}(\mathcal{B})$, for a small weakly finitely cocomplete category $\mathcal{B}$.*

Proof (1) $\Rightarrow$ (2) follows from theorem 4.11 and the first argument in the proof of 4.13, both in [2]. Choose $\mathcal{B}$ in that proof to be the full subcategory of finitely presentable objects in $\mathcal{K}$: the proof starts with proving that $\mathcal{B}$ is weakly finitely cocomplete.

(2) $\Rightarrow$ (1). The second argument in the proof of 4.13 in [2] is to infer, from the weak finite cocompleteness of $\mathcal{B}$ (and nothing else), that $\mathcal{K}$ is sketchable by a limit-epi sketch. Then applying II in that same proof, we conclude that $\mathcal{K}$ is weakly locally presentable, thus has products by theorem 4.11 in [2]. $\square$

Even if not explicitly mentioned in [2], the following result follows easily from arguments contained in that book:

$\boxed{WR}$ **Theorem 26** *Consider a full subcategory $\mathcal{P} \subseteq \mathcal{F}$ of a small category $\mathcal{F}$ with weak finite colimits. The following conditions are equivalent:*

1. *$\mathsf{Ind}(\mathcal{P})$ is weakly reflective in $\mathsf{Ind}(\mathcal{F})$;*
2. *$\mathsf{Ind}(\mathcal{P})$ is closed in $\mathsf{Ind}(\mathcal{F})$ under products;*
3. *$\mathcal{P}$ is weakly reflective in $\mathcal{F}$.*

Proof The Kan extension along the full inclusion $\mathcal{P} \rightarrowtail \mathcal{F}$ yields the full inclusion $J\colon \mathsf{Ind}(\mathcal{P}) \rightarrowtail \mathsf{Ind}(\mathcal{F})$. More precisely, if $P \in \mathsf{Ind}(\mathcal{P})$ is written as a canonical filtered colimit $P = \mathsf{colim}_i\, \mathcal{P}(-, A_i)$, then $J(P) = \mathsf{colim}_i\, \mathcal{F}(-, A_i)$.

(1) $\Rightarrow$ (2) follows from theorem 4.8 in [2] and (2) $\Rightarrow$ (3), from the first part of the proof of $(ii) \Rightarrow (i)$ in that same theorem.

(3) $\Rightarrow$ (2). For each $M \in \mathcal{F}$, consider a weak reflection $\eta_M\colon M \longrightarrow M^*$. Observe first that $\mathsf{Ind}(\mathcal{P})$ is the injectivity class for these morphisms η_M, that is, given $X \in \mathsf{Ind}(\mathcal{F})$, $X \in \mathsf{Ind}(\mathcal{P})$ precisely when every morphism $\alpha\colon M \longrightarrow X$

factors through η_M. In fact, this argument is the last part of the proof of $(ii) \Rightarrow (i)$ in theorem 4.8 in [2]; one can then conclude by the implication $(i) \Rightarrow (ii)$ of the same theorem.

(2) $\Rightarrow$ (1). By 2.53 in [2], $\mathsf{Ind}(\mathcal{P})$ is cone-reflective in $\mathsf{Ind}(\mathcal{F})$, thus weakly reflective by remark 4.5.(2) still in [2]. $\square$

Theorem 26 was recently proved in [13] in a special case of modules over Artin algebras. We particularize now this theorem 26 to the case of a variety:

WRV **Theorem 27** *For a variety $\mathcal{V}$, the following conditions are equivalent:*

1. *$\mathcal{P}$ is weakly finitely cocomplete;*
2. *$\mathcal{P}$ is weakly reflective in $\mathcal{F}$;*
3. *$\mathsf{Flat}(\mathcal{V})$ is weakly reflective in $\mathcal{V}$;*
4. *$\mathsf{Flat}(\mathcal{V})$ is closed under products in $\mathcal{V}$.*

Proof In the case of a variety, we have

$$\mathsf{Flat}(\mathcal{V}) = \mathsf{Ind}(\mathcal{P}), \quad \mathcal{V} = \mathsf{Ind}(\mathcal{F})$$

with $\mathcal{F}$ finitely cocomplete. Thus the equivalence of the last three conditions follows at once from theorem 26. Given a finite diagram in $\mathcal{P}$, a weak reflection in $\mathcal{P}$ of its colimit in $\mathcal{F}$ yields a weak colimit in $\mathcal{F}$; thus (2) implies (1). Conversely every object $M \in \mathcal{F}$ is the coequalizer in $\mathcal{F}$ of two morphisms in $\mathcal{P}$; a weak coequalizer of these morphisms in $\mathcal{P}$ yields a weak reflection of M in $\mathcal{P}$ (proof analogous to that in theorem 22). $\square$

ex2 **Examples 28**

1. Let $\mathcal{V}$ be the variety of all left R-modules. Then $\mathsf{Flat}(\mathcal{V})$ is weakly reflective iff R is right coherent (see [9], 3.2.24).
2. Let $\mathcal{C}$ be a small category and $\mathcal{V} = \mathsf{Set}^{\mathcal{C}^{op}}$. Then $\mathsf{Flat}(\mathcal{V})$ is weakly reflective in $\mathcal{V}$ iff $\mathsf{Fam}(\mathcal{C})$ has weak finite colimits. This is equivalent to $\mathcal{C}$ having weak finite connected colimits. $\square$

Finally let us mention an amazing result:

LWC **Proposition 29** *If a small category $\mathcal{C}$ is finitely complete and weakly finitely cocomplete, it is necessarily finitely cocomplete.*

Proof The finitely presentable objects of $\mathsf{Ind}(\mathcal{C})$ are the retracts of the representables. Since $\mathcal{C}$ is Cauchy complete (it is finitely complete), $\mathcal{C}$ is thus equivalent to the category of finitely presentable objects in $\mathsf{Ind}(\mathcal{C})$.

By proposition 25, $\mathsf{Ind}(\mathcal{C})$ has products. But since $\mathcal{C}$ has finite limits, $\mathsf{Ind}(\mathcal{C})$ has finite limits as well (see [4]). Since it has all products, $\mathsf{Ind}(\mathcal{C})$ is complete, thus is locally finitely presentable. Therefore the full subcategory of finitely presentable objects of $\mathsf{Ind}(\mathcal{C})$, which is equivalent to $\mathcal{C}$, has finite colimits. $\square$

VWC **Corollary 30** *For a variety $\mathcal{V}$, the following conditions are equivalent:*

1. *$\mathcal{F}$ is finitely complete;*
2. *$\mathcal{F}$ is finitely weakly complete.*

Proof By proposition 29 applied to the dual of $\mathcal{F}$. $\square$

7 Examples and counter-examples

As far as the purity of subobjects is concerned, let us consider the following conditions

1. the von Neumann axiom in $\mathcal{P}$,
2. the von Neumann axiom in $\mathcal{F}$,
3. every subobject is pure,
4. every algebra is flat,

which are known to be equivalent in the abelian case (theorem 21). We have already proved, for an arbitrary variety, the implications corresponding to the solid arrows in diagram 2 as well as various equivalences with one of the properties just listed. We shall now disprove all the other implications, corresponding to dotted arrows in diagram 2.

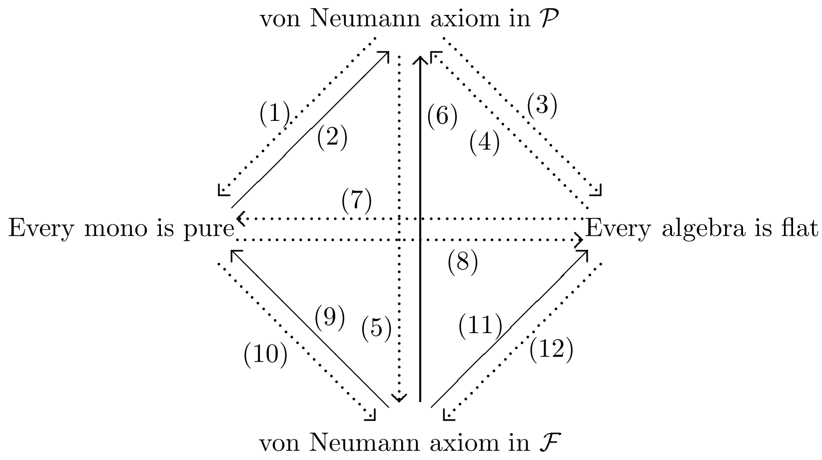

Diagram 2

The first example which will fit our needs is an obvious one.

40 **Example 31** *When $\mathcal{V}$ is the variety of sets, the following properties hold:*

1. *$\mathcal{P} = \mathcal{F}$ is the category of finite sets;*

2. *every object is flat;*

3. *a monomorphism $s\colon X \rightarrowtail Y$ is pure iff X is non-empty or Y is empty; thus not every monomorphism is pure;*

4. *a morphism $f\colon X \longrightarrow Y$ admits a morphism $g\colon Y \longrightarrow X$ such that $fgf = f$ iff X is non-empty or Y is empty; thus the $\mathcal{P} = \mathcal{F}$ does not satisfy the von Neumann axiom.* $\square$

Our second example presents some "duality" with the first one, in the sense that the corresponding categories of finitely presentable objects are dual of each other (see [5]).

$\boxed{38}$ **Example 32** *When $\mathcal{V}$ is the variety of boolean algebras, the following properties hold;*

1. *$\mathcal{P}$ is the dual of the category of non-empty finite sets;*

2. *$\mathcal{F}$ is the dual of the category of finite sets;*

3. *$\mathcal{P}$ satisfies the von Neumann axiom, but $\mathcal{F}$ does not;*

4. *every monomorphism is pure;*

5. *not every boolean algebra is flat.*

Proof Let us first review some well-known facts about boolean algebras (see, e.g., [5]). The following functors between the categories of finite boolean algebras and finite sets are contravariant inverse equivalences:

$$\mathsf{FinBool} \underset{\mathsf{Bool}(-,\{0,1\})}{\overset{\mathsf{Set}(-,\{0,1\})}{\leftrightarrows}} \mathsf{FinSet};$$

this is the "finite part" of the Stone duality.

As a consequence, for each $n \in \mathbb{N}$, the free boolean algebra Fn on n generators satisfies

$$Fn \cong \mathsf{Set}\big(\mathsf{Bool}(Fn,\{0,1\}),\{0,1\}\big) \cong \mathsf{Set}\big(\mathsf{Set}(n,\{0,1\}),\{0,1\}\big) \cong 2^{2^n}.$$

Every finitely presentable boolean algebra is a quotient of a free algebra 2^{2^n}, thus is finite. But by the equivalence above, the finite boolean algebras are exactly the algebras 2^n, for all finite sets n. Since every finite set n can obviously be presented as the equalizer of two arrows of the form $2^k \rightrightarrows 2^l$, every finite boolean algebra is in fact finitely presentable. Since a finite set is injective iff it is non empty, a finite boolean algebra 2^n is projective iff $n \neq 0$. This takes already care of conditions 1 and 2 of the statement.

As observed in example 31, the category of non-empty finite sets satisfies the von Neumann axiom, but the category of finite sets does not. Since this axiom is selfdual, this proves condition 3.

By theorem 4 and condition 2 of the present example, the purity of every morphism is equivalent to the fact that every morphism of finite sets factors as a split epimorphism followed by a regular monomorphism. But every epimorphism of finite sets is split and every monomorphism of finite sets is regular.

Finally every boolean algebra in $\mathcal{P}$ is a 2^n with $n \neq 0$, thus is such that $0 \neq 1$. Every filtered colimit of such algebras has the same property. This proves that the terminal boolean algebra $\{0 = 1\}$ is not flat. $\square$

$\boxed{45}$ **Example 33** *Let $\mathcal{V}$ be the theory of pointed left M-sets on the monoid*

$$M = \{1, \alpha, \beta\}, \quad \alpha\alpha = \alpha\beta = \alpha, \quad \beta\beta = \beta\alpha = \beta.$$

The following properties hold:

1. *$\mathcal{P}$ satisfies the von Neumann axiom, but $\mathcal{F}$ does not;*
2. *not every monomorphism is pure;*
3. *not every algebra is flat.*

Proof Let us write e for the base point of an algebra. Every algebra contains thus the constants e, αe and βe. Given $n \in \mathbb{N}$, the free algebra Fn on n-generators $x_1, \ldots, x_n$ is constituted of the following elements

$$\begin{array}{lll} e & \alpha e & \beta e \\ x_1 & \alpha x_1 & \beta x_1 \\ \vdots & \vdots & \vdots \\ x_n & \alpha x_n & \beta x_n \end{array}$$

and these elements are all distinct.

Up to renumbering of the generators $x_1, \ldots, x_n$, a subobject $X \subseteq Fn$ has thus necessarily the form

$$\begin{array}{lll} e & \alpha e & \beta e \\ x_1 & \alpha x_1 & \beta x_1 \\ \vdots & \vdots & \vdots \\ x_l & \alpha x_l & \beta x_l \\ & \alpha x_{l+1} & \beta x_{l+1} \\ & \vdots & \vdots \\ & \alpha x_m & \beta x_m \end{array}$$

for some natural numbers $0 \leq l \leq m \leq n$. Such an object is easily seen to be a retract in $\mathcal{V}$, with retraction $r \colon Fn \longrightarrow X$ determined by

$$\begin{array}{rcll} r(e) & = & e; & \\ r(x_i) & = & x_i & \text{for } 1 \leq i \leq l; \\ r(x_i) & = & \alpha x_i & \text{for } l+1 \leq i \leq m; \\ r(x_i) & = & e & \text{for } m+1 \leq i \leq n. \end{array}$$

Of course one could have chosen equivalently $r(x_i) = \beta x_i$ in the third case. It is routine to check that r is an homomorphism presenting X as a retract of Fn.

We know that $\mathcal{P}$ is the Cauchy completion of the category of free algebras on finitely many generators. Thus every object $A \in \mathcal{P}$ is a retract of some Fn, that is one of the subobjects described above. But a subobject $s\colon X \rightarrowtail A$ of an object $A \in \mathcal{P}$ is then also a subobject of some Fn, thus as we have seen a retract of this Fn with a corresponding retraction r.

$$\begin{tikzcd} X \arrow[r, tail, "s"'] & A \arrow[r, tail, "i"'] & Fn \arrow[ll, bend right, "r"'] \end{tikzcd}$$

The equality $ris = \mathsf{id}_X$ shows that X is a retract of A. By theorem 21, this implies the von Neumann axiom in $\mathcal{P}$.

Now let us consider the free algebra $F1$ on one generator x and let us perform its quotient by the relation $\alpha x \approx \beta x$. This quotient is thus finitely presentable and is simply

$$Q = \{e,\ \alpha e,\ \beta e,\ x,\ \alpha x = \beta x\}.$$

As subobject of Q, we consider the initial algebra

$$0 = \{e,\ \alpha e,\ \beta e\}.$$

The inclusion $0 \hookrightarrow Q$ does not admit a retraction, because Q contains an element $y = \alpha x = \beta x$ such that $\alpha y = \beta y$, while 0 does not. The initial algebra 0 is free on the empty set, thus finitely generated. So by theorem 4, not every monomorphism is pure in $\mathcal{V}$.

Finally in the description of the objects of $\mathcal{P}$ given above, notice that we have always $\alpha e \neq \beta e$. Thus the same inequality holds in every filtered colimit of such objects, that is in every flat algebra. So the terminal algebra $1 = \{e = \alpha e = \beta e\}$ is not flat. By theorem 18, $\mathcal{F}$ does not satisfy the von Neumann axiom. $\square$

We are now able to justify the content of diagram 2:

1	By example 33.	7	By example 31.
2	By theorem 17.	8	By example 32.
3	By example 32 or 33.	9	By theorem 18.
4	By example 31.	10	By example 32.
5	By example 32 or 33.	11	By theorem 18.
6	Obvious.	12	By example 31.

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